# Semi-implicit WENO schemes

B. Crnković
N. Črnjarić-Žic
S. Maćešić

Faculty of Engineering
University of Rijeka
Vukovarska 58
51000 Rijeka

## References


- **A. Harten**: On a Class of High Resolution Total-Variation-Stable Finite-Difference Schemes, SIAM J. Numer. Anal. **21(1)** (1984)
- **S. Vuković, L. Sopta**: Upwind schemes with exact conservation property for one-dimensional open channel flow equations, SIAM J. Sci. Comp. 24(5), 1630-1649 (2003)
- **N. Črnjarić-Žic ,S. Vuković, L. Sopta**:Balanced finite volume WENO and central WENO schemes for the shallow water and the open-channel flow equations, J. Comput.Phys. 200, 512-548 (2004)
- **Y. Xing, C.W. Shu**: High order well–balanced finite volume WENO schemes and discontinuous Galerkin methods for a class of hyperbolic systems with source term, JCP, 214 (2006), 567-598
- **S. Noelle, N. Pankratz, G. Puppo, J.R. Natvig**: Well–balanced finite volume schemes of arbitrary order of accuracy for shallow water flows, JCP, 213 (2006), 474-499

## 1D Hyperbolic balance law:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u}, x)}{\partial x} = \mathbf{g}(\mathbf{u}, x), \quad x \in \mathbb{R}, \ t > 0.$$

## Numerical schemes:

**Explicit:**

$$\mathbf{u}_i^{n+1} = \mathbf{u}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{f}_{i+1/2}^n - \mathbf{f}_{i-1/2}^n\right) + \Delta t \mathbf{g}_i^n$$

**Implicit:**

$$\mathbf{u}_i^{n+1} = \mathbf{u}_i^n - \frac{\Delta t}{\Delta x}\left(\mathbf{f}_{i+1/2}^{n+1} - \mathbf{f}_{i-1/2}^{n+1}\right) + \Delta t \mathbf{g}_i^{n+1}$$

**Semi-implicit:**

$$\begin{aligned}\mathbf{u}_i^{n+1} = \mathbf{u}_i^n &- (1-\mu)\left[\frac{\Delta t}{\Delta x}\left(\mathbf{f}_{i+1/2}^n - \mathbf{f}_{i-1/2}^n\right) - \Delta t \mathbf{g}_i^n\right] \\ &- \mu\left[\frac{\Delta t}{\Delta x}\left(\mathbf{f}_{i+1/2}^{n+1} - \mathbf{f}_{i-1/2}^{n+1}\right) - \Delta t \mathbf{g}_i^{n+1}\right]\end{aligned}$$

## Flux:

$$\begin{aligned}\mathbf{f}_{i+1/2} &= \frac{1}{2}\left(\mathbf{f}(\mathbf{u}^{+}_{i+1/2}) + \mathbf{f}(\mathbf{u}^{-}_{i+1/2})\right) - \frac{1}{2}\left|\mathbf{A}_{i+1/2}\right|\left(\mathbf{u}^{+}_{i+1/2} - \mathbf{u}^{-}_{i+1/2}\right)\\ &- \frac{1}{2}\mathbf{A}^{-1}_{i+1/2}\left|\mathbf{A}_{i+1/2}\right|\mathbf{v}_{i+1/2}\Delta x\end{aligned}$$

## Source:

$$\begin{aligned}\mathbf{g}_i = \mathbf{g}^{-}_{i+1/2} + \mathbf{g}^{+}_{i-1/2} &= \frac{1}{2}\left(\mathbf{I} - \mathbf{A}^{-1}_{i+1/2}\left|\mathbf{A}_{i+1/2}\right|\right)\mathbf{g}_{i+1/2}\\ &+ \frac{1}{2}\left(\mathbf{I} + \mathbf{A}^{-1}_{i-1/2}\left|\mathbf{A}_{i-1/2}\right|\right)\mathbf{g}_{i-1/2}\end{aligned}$$

$$\mathbf{A}(\mathbf{u}) = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}, \quad \mathbf{A} = \mathbf{R}\mathbf{\Lambda}\mathbf{R}^{-1}, \quad \mathbf{v}(\mathbf{u}) = \frac{\partial \mathbf{f}}{\partial x}\Big|_{\mathbf{u}=const.}$$

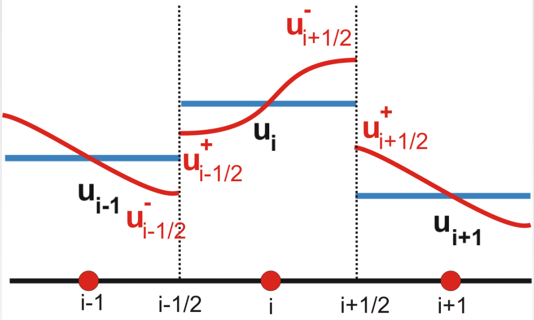



Figure: Approximate Riemann problem solver

## Roe Average

$$
\begin{aligned}
\mathbf{f}(\mathbf{u}^+_{\mathbf{i+1/2}}) - \mathbf{f}(\mathbf{u}^-_{\mathbf{i+1/2}}) &= \mathbf{A}(\mathbf{u}_{\mathbf{i+1/2}})(\mathbf{u}^+_{\mathbf{i+1/2}} - \mathbf{u}^-_{\mathbf{i+1/2}}) + \mathbf{v}(\mathbf{u}_{\mathbf{i+1/2}})\Delta x \\
\mathbf{u}_{i+1/2} &= \mathbf{u}(\mathbf{u}^-_{\mathbf{i+1/2}}, \mathbf{u}^+_{\mathbf{i+1/2}})
\end{aligned}
$$

## Taylor expansion

In order to drop the time-index from $(n+1)$ to $n$, vectors $\mathbf{f}^{n+1}$, $\mathbf{v}^{n+1}$ and $\mathbf{g}^{n+1}$ are linearized using Taylor expansion:

$$\begin{aligned}
\mathbf{f}_i^{n+1} &\approx \mathbf{f}_i^n + \mathbf{J}_f^n \left(\mathbf{u}_i^{n+1} - \mathbf{u}_i^n\right), \\
\mathbf{v}_{i+1/2}^{n+1} &\approx \mathbf{v}_{i+1/2}^n + \mathbf{J}_v^n \left(\mathbf{u}_{i+1/2}^{n+1} - \mathbf{u}_{i+1/2}^n\right), \\
\mathbf{g}_{i+1/2}^{n+1} &\approx \mathbf{g}_{i+1/2}^n + \mathbf{J}_g^n \left(\mathbf{u}_{i+1/2}^{n+1} - \mathbf{u}_{i+1/2}^n\right),
\end{aligned}$$

## Semi-implicit scheme ⇒ block $n$-diagonal linear system

$$(\mathbf{I} + \mu \mathbf{K}(\mathbf{u}^\mathbf{n})) \cdot \left(\mathbf{u}^{n+1} - \mathbf{u}^n\right) = \mathbf{d}(\mathbf{u}^\mathbf{n}),$$

## Roe scheme (**Q** scheme)

$$O(\Delta t) + O(\Delta x)$$

$$\begin{aligned} \mathbf{u}^{-}_{\mathbf{i+1/2}} &= \mathbf{u}_{\mathbf{i}} \\ \mathbf{u}^{+}_{\mathbf{i+1/2}} &= \mathbf{u}_{\mathbf{i+1}} \end{aligned}$$

## Flux and Source

$$\begin{aligned} \mathbf{f}_{i+1/2} &= \frac{1}{2}\left(\mathbf{f}_i + \mathbf{f}_{i+1}\right) - \frac{1}{2}\mathbf{R}_{i+1/2}\left|\mathbf{\Lambda}_{i+1/2}\right|\mathbf{R}^{-1}_{i+1/2}\left(\mathbf{u}_{i+1} - \mathbf{u}_i\right) \\ &- \frac{1}{2}\mathbf{R}_{i+1/2}\mathbf{\Lambda}^{-1}_{i+1/2}\left|\mathbf{\Lambda}_{i+1/2}\right|\mathbf{R}^{-1}_{i+1/2}\mathbf{V}_{i+1/2}\Delta x \\ \mathbf{g}_i &= \frac{1}{2}\left(\mathbf{I} - \mathbf{R}_{i+1/2}\mathbf{\Lambda}^{-1}_{i+1/2}\left|\mathbf{\Lambda}_{i+1/2}\right|\mathbf{R}^{-1}_{i+1/2}\right)\mathbf{g}_{i+1/2} \\ &+ \frac{1}{2}\left(\mathbf{I} + \mathbf{R}_{i-1/2}\mathbf{\Lambda}^{-1}_{i-1/2}\left|\mathbf{\Lambda}_{i-1/2}\right|\mathbf{R}^{-1}_{i-1/2}\right)\mathbf{g}_{i-1/2} \end{aligned}$$

## Flux limited scheme (**FL** scheme)

$$O(\Delta t) + O(\Delta x^2)$$

$$\begin{aligned}
\mathbf{u}^-_{\mathbf{i+1/2}} &= \mathbf{u_i} \\
\mathbf{u}^+_{\mathbf{i+1/2}} &= \mathbf{u_{i+1}}
\end{aligned}$$

## Flux and Source

$$\begin{aligned}
\mathbf{f}_{i+1/2} &= \frac{1}{2}\left(\mathbf{f}_i + \mathbf{f}_{i+1}\right) - \frac{1}{2}\mathbf{R}_{i+1/2}\left|\mathbf{\Lambda}_{i+1/2}\right|\mathbf{L}_{i+1/2}\mathbf{R}^{-1}_{i+1/2}\left(\mathbf{u}_{i+1} - \mathbf{u}_i\right) \\
&- \frac{1}{2}\mathbf{R}_{i+1/2}\mathbf{\Lambda}^{-1}_{i+1/2}\left|\mathbf{\Lambda}_{i+1/2}\right|\mathbf{L}_{i+1/2}\mathbf{R}^{-1}_{i+1/2}\mathbf{V}_{i+1/2}\Delta x \\
\mathbf{g}_i &= \frac{1}{2}\left(\mathbf{I} - \mathbf{R}_{i+1/2}\mathbf{\Lambda}^{-1}_{i+1/2}\left|\mathbf{\Lambda}_{i+1/2}\right|\mathbf{L}_{i+1/2}\mathbf{R}^{-1}_{i+1/2}\right)\mathbf{g}_{i+1/2} \\
&+ \frac{1}{2}\left(\mathbf{I} + \mathbf{R}_{i-1/2}\mathbf{\Lambda}^{-1}_{i-1/2}\left|\mathbf{\Lambda}_{i-1/2}\right|\mathbf{L}_{i-1/2}\mathbf{R}^{-1}_{i-1/2}\right)\mathbf{g}_{i-1/2}
\end{aligned}$$

### Roe

$$\mathbf{L}_{i+1/2} = \mathbf{I}$$

### Lax-Wendroff

$$\mathbf{L}_{i+1/2} = |\mathbf{\Lambda}_{i+1/2}| \frac{\Delta t}{\Delta x}$$

### Flux limited

$$\mathbf{L}_{i+1/2} = \mathbf{I} - \left( \mathbf{\Phi}_{i+1/2} \left( \mathbf{I} - \left| \mathbf{\Lambda}_{i+1/2} \right| \frac{\Delta t}{\Delta x} \right) \right)$$

### Modified flux limited

$$\mathbf{L}_{i+1/2} = \mathbf{I} - \left( \mathbf{\Phi}_{i+1/2} \left( \mathbf{I} - diag\left( min\left(1, \left| \lambda_{i+1/2,p} \right| \frac{\Delta t}{\Delta x} \right) \right) \right) \right)$$

# Weighted Essentially NonOscillatory schemes

## WENO($s$,$r$)

- uses a $s$-stage TVD Runge-Kuta time integration,
- high order polynomial reconstruction in the spatial domain.
- Formal accuracy: $O(\Delta t^s) + O(\Delta x^{2r+1})$
- $C_{cfl} = \frac{\Delta t}{\Delta x} max(\lambda) \leq 1$

## Implicit WENO($1$,$r$)

- uses a backward Euler time integration
- Formal accuracy: $O(\Delta t) + O(\Delta x^{2r+1})$
- At each time step one needs to solve a block $(2r+3)$-diagonal linear system.
- Unconditionally stable.

## Each reconstructed state is a convex combination of averaged cell states

$$
\begin{aligned}
\mathbf{u}^{-}_{\mathbf{i+1/2}}[p] &= \sum_{k=0}^{r}\sum_{j=0}^{r}\omega_k^{-}(\mathbf{u}[p])\cdot a^{-}_{k,j}\cdot \mathbf{u}_{\mathbf{i-r+k+j}}[p]\\
\mathbf{u}^{+}_{\mathbf{i+1/2}}[p] &= \sum_{k=1}^{r+1}\sum_{j=0}^{r}\omega_k^{+}(\mathbf{u}[p])\cdot a^{+}_{k,j}\cdot \mathbf{u}_{\mathbf{i-r+k+j}}[p]
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=0}^{r}\omega_k^{-}(\mathbf{u}[p]) &= 1, \ \sum_{k=0}^{r}\sum_{j=0}^{r} a^{-}_{k,j} = 1\\
\sum_{k=1}^{r+1}\omega_k^{+}(\mathbf{u}[p]) &= 1, \ \sum_{k=1}^{r+1}\sum_{j=0}^{r} a^{+}_{k,j} = 1
\end{aligned}
$$

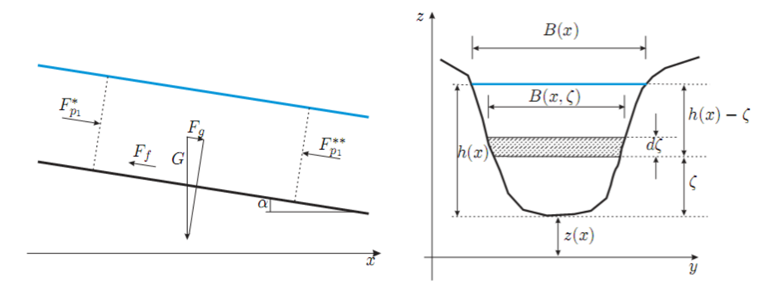



Figure: Control volume

## Open Channel

$$\mathbf{u} = \begin{bmatrix} A \\ Q \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} Q \\ \frac{Q^2}{A} + gI_1 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} 0 \\ gI_2 + gA\left(S_b - S_f\right) \end{bmatrix}.$$

- $t$ denotes time
- $x$ is the horizontal distance along the channel
- $A = A(x, t)$ is the wetted cross-sectional area
- $Q = Q(x, t)$ represents discharge
- $I_1 = \int_0^h (h - \eta) B(\eta, x) d\eta$ models the hydrostatic pressure force
- $I_2 = \int_0^h (h - \eta) \frac{\partial B(\eta, x)}{\partial x} d\eta$ the pressure force due to longitudinal channel width variation.
- $S_f = m^2 Q \left| Q \right| P^{4/3} A^{-10/3}$ is the friction term
- $S_b = -\frac{dz}{dx}$ is the bed slope
- $B = B(h, x)$ is the channel width
- $m = m(x)$ is the Manning's roughness coefficient
- $P = P(A, x)$ is the perimeter
- $z = z(x)$ is the channel bed
- $h = h(A, x)$ is the water depth

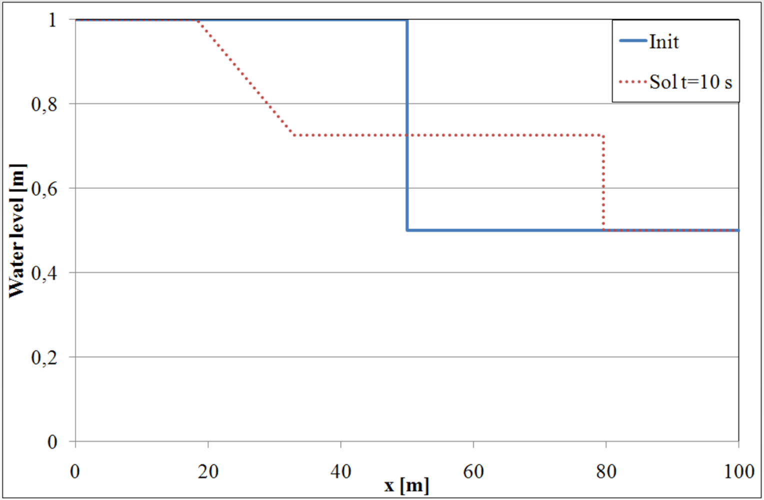



Figure: Initial and exact solution to a Dam break problem at $t$=10 $s$

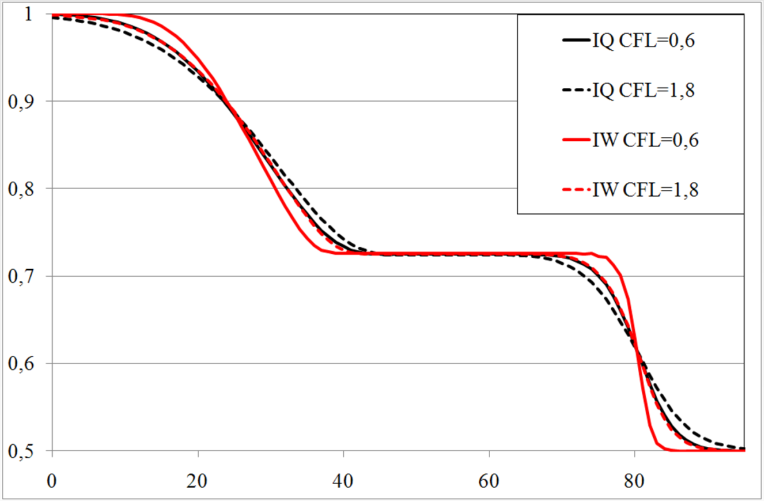



Figure: Q and WENO(1,3) implicit scheme with different CFL numbers

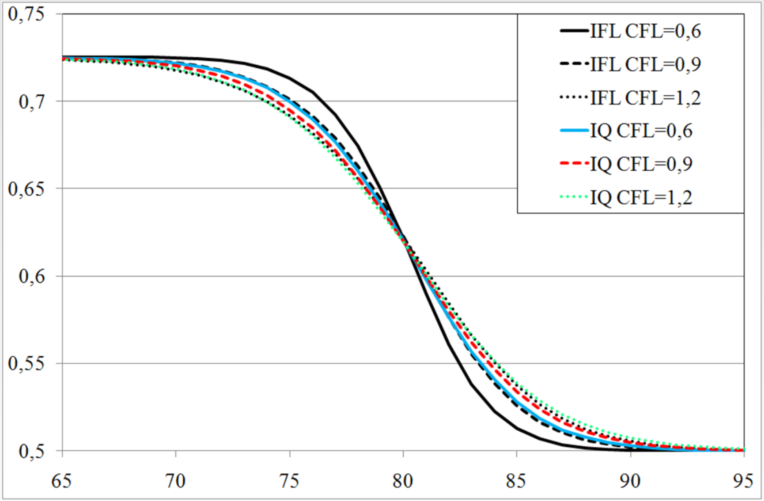



Figure: Q and Flux limited implicit scheme

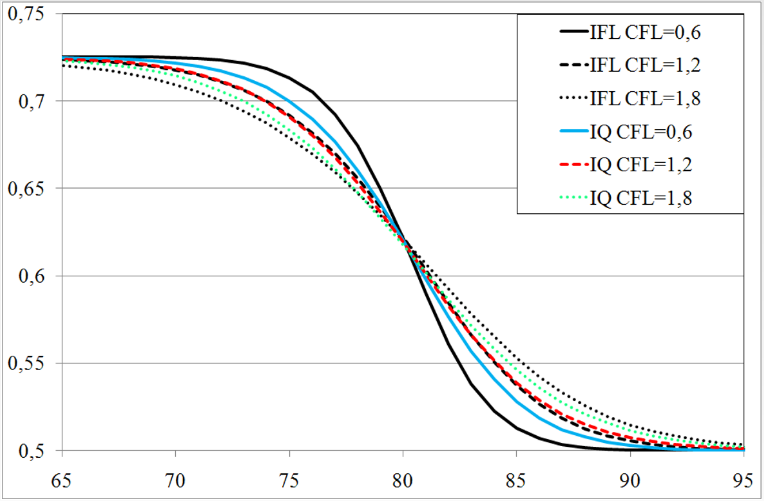



Figure: Q and Flux limited implicit scheme

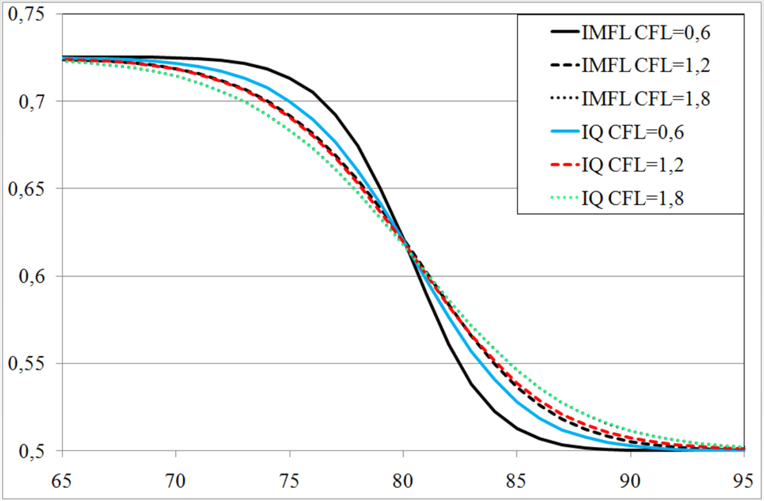



Figure: Q and modified Flux limited implicit scheme

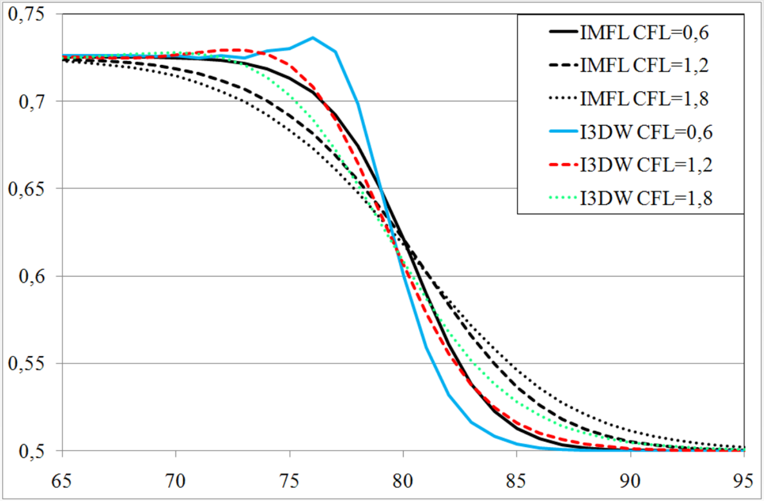



Figure: Modified Flux limited and block three-diagonal component wise WENO(1,3) implicit scheme

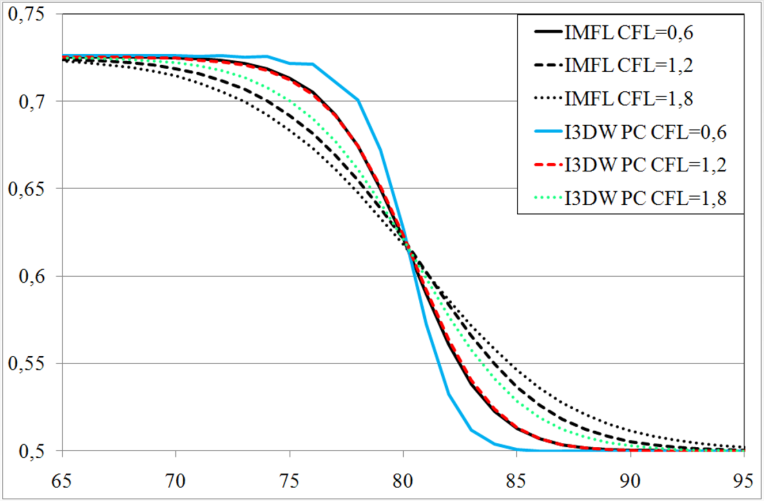



Figure: Modified Flux limited and block three-diagonal component wise Predictor-Corrector WENO(1,3) implicit scheme

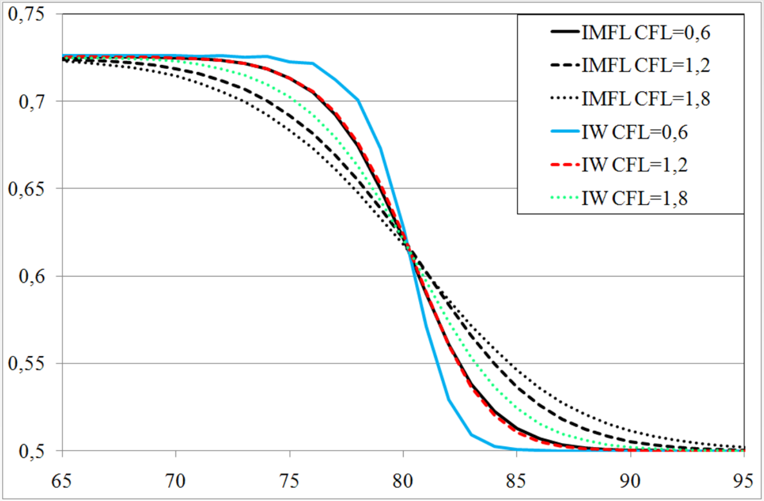



Figure: Modified Flux limited and block component wise nine-diagonal WENO(1,3) implicit scheme

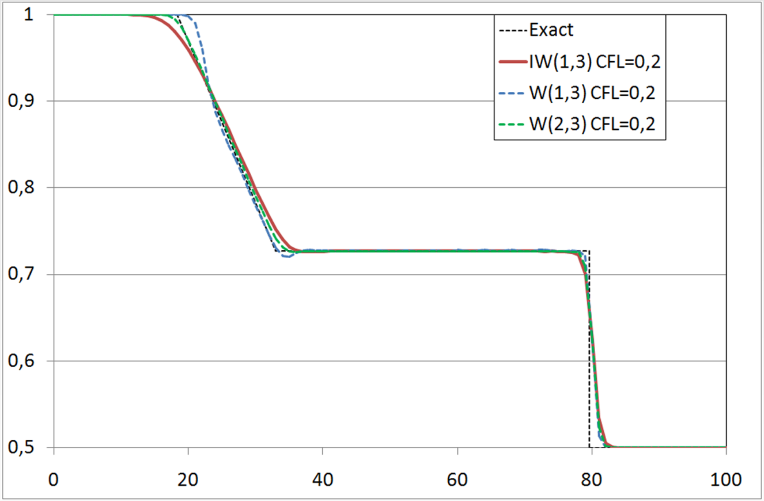



Figure: Finite volume characteristic wise WENO(1,3) and implicit WENO(1,3) scheme

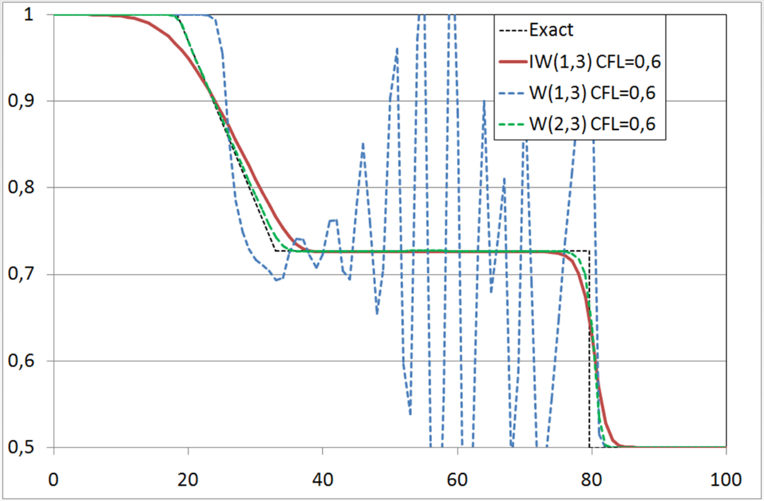
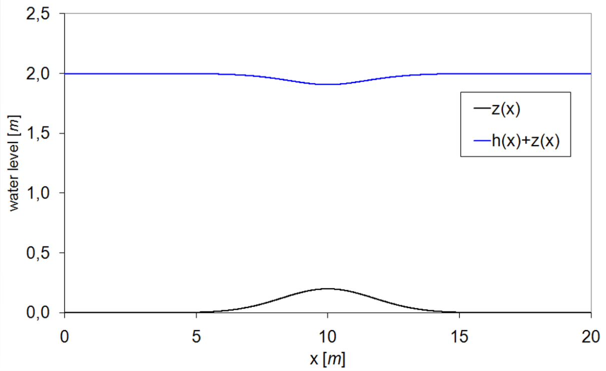



Figure: Finite volume characteristic wise WENO(1,3) and implicit WENO(1,3) scheme

## Steady-state test problem used for testing the convergence

- riverbed bottom is supposed to be given with a smooth function $z(x) = 0,2e^{-\frac{4}{25}(x-10)^2}$, where $x \in [0;20]$
- steady subcritical flow with a constant discharge equal $Q = 4,42\ m^2/s$



# IMPLICIT WENO(1,3)

## $C_{cfl}$=0,5 $t$=0,1 $s$

| N | $\| \|_1$ | Order | $\| \|_\infty$ | Order |
|---|---|---|---|---|
| 20 | $3,3 \cdot 10^{-4}$ | | $1,1 \cdot 10^{-4}$ | |
| 40 | $4,4 \cdot 10^{-5}$ | 2,9 | $1,6 \cdot 10^{-5}$ | 2,7 |
| 80 | $9,3 \cdot 10^{-6}$ | 2,3 | $3,4 \cdot 10^{-6}$ | 2,3 |
| 160 | $2,2 \cdot 10^{-6}$ | 2,1 | $8,1 \cdot 10^{-7}$ | 2,0 |

## $C_{cfl}$=1 $t$=0,1 $s$

| N | $\| \|_1$ | Order | $\| \|_\infty$ | Order |
|---|---|---|---|---|
| 20 | $3,0 \cdot 10^{-4}$ | | $9,9 \cdot 10^{-5}$ | |
| 40 | $4,6 \cdot 10^{-5}$ | 2,7 | $1,6 \cdot 10^{-5}$ | 2,6 |
| 80 | $9,8 \cdot 10^{-6}$ | 2,2 | $3,5 \cdot 10^{-6}$ | 2,2 |
| 160 | $2,3 \cdot 10^{-6}$ | 2,1 | $8,4 \cdot 10^{-7}$ | 2,1 |

# IMPLICIT WENO(1,2)

$C_{cfl}$=0.01 $t$=0,01 $s$

| N | $\| \|_1$ | Order | $\| \|_\infty$ | Order |
|---|---|---|---|---|
| 20 | $3{,}1\cdot10^{-4}$ | | $9{,}9\cdot10^{-5}$ | |
| 40 | $7{,}0\cdot10^{-5}$ | 2,1 | $1{,}6\cdot10^{-5}$ | 2,2 |
| 80 | $1{,}2\cdot10^{-5}$ | 2,6 | $3{,}5\cdot10^{-6}$ | 2,3 |
| 160 | $2{,}4\cdot10^{-6}$ | 2,3 | $8{,}4\cdot10^{-7}$ | 2,4 |

$C_{cfl}$=0.01; $t$=0,01;Romberg $O(\Delta x^4)$ integration of the source term

| N | $\| \|_1$ | Order | $\| \|_\infty$ | Order |
|---|---|---|---|---|
| 20 | $6{,}2\cdot10^{-5}$ | | $9{,}9\cdot10^{-5}$ | |
| 40 | $7{,}2\cdot10^{-6}$ | 3,1 | $1{,}6\cdot10^{-6}$ | 3,3 |
| 80 | $4{,}4\cdot10^{-7}$ | 4,0 | $3{,}5\cdot10^{-7}$ | 3,7 |
| 160 | $3{,}5\cdot10^{-8}$ | 3,6 | $8{,}4\cdot10^{-8}$ | 3,4 |