STEREOGRAPHIC MAP PROJECTION OF CROATIA

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ABSTRACT: This paper describes an application of the stereographic map projection for the territory of the Republic of Croatia. The stereographic map projection for Croatia has not been explored yet. When choosing a map projection for the region on the Earth's surface, its size and shape are among the first criteria. The Republic of Croatia has a small area (~0.02% of the area of the Earth ellipsoid) with a specific shape resembling the letter "C". Applications of the stereographic map projection can be found in historical and modern cartography. The stereographic projection of the sphere and its properties are well-known. A rotational ellipsoid is often chosen as an Earth model in cartography and geodesy. The stereographic projection of the rotational ellipsoid is not unique. Different definitions and solutions can be found in literature. Conformality is often preserved due to its importance to geodesy. First, an overview of approaches to stereographic map projection of rotational ellipsoid found in literature is given. Then, an approach of conformal mapping of the rotational ellipsoid onto the sphere and stereographic projection of the sphere onto the plane are chosen. We give equations, an analysis of linear deformations and an analysis of optimal parameters according to Airy/Jordan criteria for the region of the Republic of Croatia. Further investigation for finding optimal parameters is also proposed.

Keywords: map projection, stereographic projection, rotational ellipsoid, Croatia

1. INTRODUCTION

The stereographic projection of the sphere is considered as one of the oldest known projections, dating from ancient Egypt. The name was given by d'Aiguillon in place of the earlier name *planisphere* in 1613 [13]. Its two distinct properties are conformality (angle preserving) and circle preserving. Its spherical form is used for mapping the Earth's sphere, celestial charting, mathematics, crystallo graphy, etc.

The stereographic projection of rotational ellipsoid dates from the 19th century. In [9], we can find a geometrical proof that (geometrical) stereographic projection of any planar section of ellipsoid onto the projection plane is a circle. V. V. Kavrayskiy [6] mentioned that the same problem was given to students by Legendre in 1805. Kavrayskiy also proposed the definition of the stereographic projection of the ellipsoid as any generalisation which, for the spherical case, gives the known stereographic projection of the sphere. O. Eggert [3] states that C. F. Gauss was probably the first who employed the stereographic projection of ellipsoid in geodesy. Gauss used conformal mapping of the ellipsoid onto the sphere which is then projected onto the plane using the stereographic projection. Two variants exist, differing in the selection of the radius of the sphere. L. Krüger [7] derives equations based on that idea. M. H. Roussilhe [11] gives another variant in which the chosen meridian is mapped in a stereographical manner as if it is meridian on the sphere, and the second constraint he used was conformality. Later, W. Hristow [5] derives new formulas based on the idea of Roussilhe and finds that formulas Roussilhe gave were not strictly conformal. Eggert [3] proposed another variant. He stated that stereographic projection of the sphere can be defined in two ways. One definition is that it is a perspective projection, and the other is that it is a conformal azimuthal projection. He used the second definition for deriving the stereographic projection of the ellipsoid.

Today, the stereographic projection of the ellipsoid is used for mapping the polar regions (so-called UPS, Universal Polar Stereographic). In that form, it is azimuthal, but not perspective [13]. It is also used or was used for large scale mapping in some countries such as The Netherlands, Romania, Poland, Hungary, etc.

Stereographic projection is sometimes used in cartography as a basis for complex-algebra polynomial transformations of the plane which yields better adaptation of linear deformations to the shape of the mapped region [2].

2. STEREOGRAPHIC PROJECTION FOR CROATIA

The stereographic projection has not been previously considered for the region of Croatia. A. Fashing [12] considered it for the area of Yugoslavia in 1924. His proposal was not accepted, instead the Gauss-Krüger map projection was the winner. Croatia was part of Yugoslavia at the time.

First we have to decide which existing variant of stereographic projection to choose or we can try to devise a new one. For the purpose of this work, we decided to use the mentioned Gauss approach, i.e. conformal mapping of the ellipsoid onto the sphere and then stereographic mapping of the sphere onto the plane. The reason is simplicity of such an approach and avoidance of power series [6].

Conformal mapping of an ellipsoid onto a sphere can be defined by following equations [3, 16]:

$$\lambda = \alpha L$$

$$\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)=\frac{1}{K}\tan^{\alpha}\left(\frac{\pi}{4}+\frac{B}{2}\right)\left(\frac{1-e\sin B}{1+e\sin B}\right)^{\frac{\alpha e}{2}} \quad (1)$$

where φ, λ are the latitude and the longitude on the sphere, B, L are the latitude and the longitude on the ellipsoid, e is the first eccentricity of the ellipsoid and α, K are parameters of that mapping. In general, a conformal mapping of the ellipsoid onto a sphere is not a function of the radius R of the sphere. The radius R of the sphere appears as third parameter when we consider a linear scale of that mapping (see equation 5).

C. F. Gauss (see e.g. [16]) gave the solution for α, K and R when the linear scale in a chosen point B_0, L_0 on the ellipsoid equals 1 and is as close to 1 as possible for other points. The constants α_0, K_0 and R_0 depending on the point B_0, L_0 of the ellipsoid are then defined as follows:

$$\alpha_0^2 = 1+\frac{e^2}{1-e^2}\cos^4 B_0$$

$$\lambda_0 = \alpha_0 L_0$$

$$\alpha_0 \sin\varphi_0 = \sin B_0$$

$$K_0 = \frac{\tan^{\alpha_0}\left(\frac{\pi}{4}+\frac{B_0}{2}\right)}{\tan\left(\frac{\pi}{4}+\frac{\varphi_0}{2}\right)}\left(\frac{1-e\sin B_0}{1+e\sin B_0}\right)^{\frac{\alpha_0 e}{2}}$$

$$M_0 = \frac{a\left(1-e^2\right)}{\left(1-e^2\sin^2 B_0\right)^{\frac{3}{2}}}$$

$$N_0 = \frac{a}{\left(1-e^2\sin^2 B_0\right)^{\frac{1}{2}}}$$

$$R_0 = \sqrt{M_0 N_0} \quad (2)$$

Finally, the stereographic projection of the sphere onto its tangential plane at the point φ_0, λ_0 can be written (with minor rearrangements of [1] or [14]):

$$x = \frac{2R_0}{k}\left(\sin\varphi\cos\varphi_0 - \sin\varphi_0\cos\varphi\cos(\lambda-\lambda_0)\right)$$

$$y = \frac{2R_0}{k}\sin(\lambda-\lambda_0)\cos\varphi \quad (3)$$

where

$$k = 1+\sin\varphi_0\sin\varphi+\cos\varphi_0\cos\varphi\cos(\lambda-\lambda_0) \quad (4)$$

with x increasing northerly, and y, easterly.

Now, the linear scale of that projection can be found as:

$$c = \frac{dS_P}{dS_S}\frac{dS_S}{dS_E}$$

where
dS_P is the differential arc length in the plane of projection;
dS_S is the differential arc length on the sphere;
dS_E is the differential arc length on the ellipsoid;
then dS_P / dS_S is the linear scale of the stereographic projection of the sphere onto the plane and dS_S / dS_E is the linear scale of the mapping of the ellipsoid onto the sphere.
According to [16]:

$$\frac{dS_S}{dS_E} = \alpha_0 \frac{R_0 \cos\varphi}{N \cos B} \tag{5}$$

where

$$N = \frac{a}{\left(1 - e^2 \sin^2 B\right)^{\frac{1}{2}}}.$$

Linear scale in the stereographic projection of the sphere onto the plane can be expressed as (with minor rearrangements of [14] or [1]):

$$\frac{dS_P}{dS_S} = \frac{2}{k}$$

where k is defined as in (4). Finally, the linear scale of the composition of the two mapping is:

$$c = \frac{2\alpha_0 R_0 \cos\varphi}{kN \cos B}.$$

Map projection parameters can be found using various criteria [4]. Airy and Jordan proposed two criteria. It is interesting that in the case of conformal map projections, both are simplified to the form:

$$E^2 = \frac{1}{A}\int_A (c-1)^2 dA$$

where A is the region over which optimum parameters are to be found. In the case of geographical regions, we approximate the integral with the following sum:

$$E^2 = \frac{1}{\Sigma \Delta A_i}\sum_{i=1}^{n}(c_i - 1)^2 \Delta A_i\,. \tag{6}$$

We will get optimal parameters of stereographic projection for the given region according to the criterion of Airy / Jordan by finding the minimum of function E.

2.1 Calculation of parameters – the first approach

In order to find optimal parameters according to the Airy/Jordan criterion for the proposed version of the stereographic map projection of the ellipsoid for the territory of Croatia, the irregular region has to be approximated with some more simple objects. We decided to take ellipsoidal quadrangles, area of which can be calculated using a closed formula. To find the optimal size of the ellipsoidal quadrangles, we applied several sizes from $\Delta B = \Delta L = 1°$ to $\Delta B = \Delta L = 1'$. Overlay of the polygon, which represents the state border of Croatia, and grid of quadrangles, gives only the quadrangles which cover the territory of Republic of Croatia.

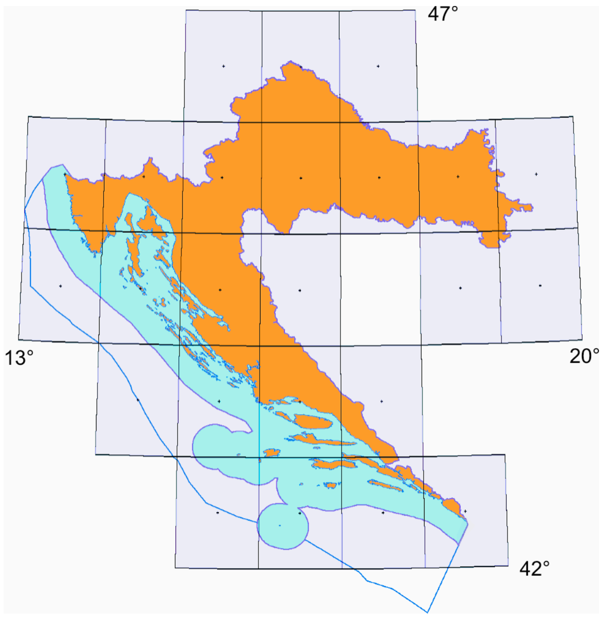

Figure 1. Approximation of the territory of Croatia with a grid of ellipsoidal quadrangles of size 1°

Specifically, the region of the Republic of Croatia is approximated with the following grids of ellipsoidal quadrangles. The first approximation uses 24 quadrangles of size $\Delta B = \Delta L = 1°$ (Fig. 1). The second approximation uses 72 quadrangles of size $\Delta B = \Delta L = 30'$ (Fig. 2), the third uses 457 quadrangles of size $\Delta B = \Delta L = 10'$ (Fig. 3). The fourth approximation uses 1653 quadrangles of size $\Delta B = \Delta L = 5'$, the fifth uses 9581 quadrangles of size $\Delta B = \Delta L = 2'$ and the sixth uses 37181 quadrangles of size $\Delta B = \Delta L = 1'$. Figures 1-3 are illustrations in a conical conformal projection.

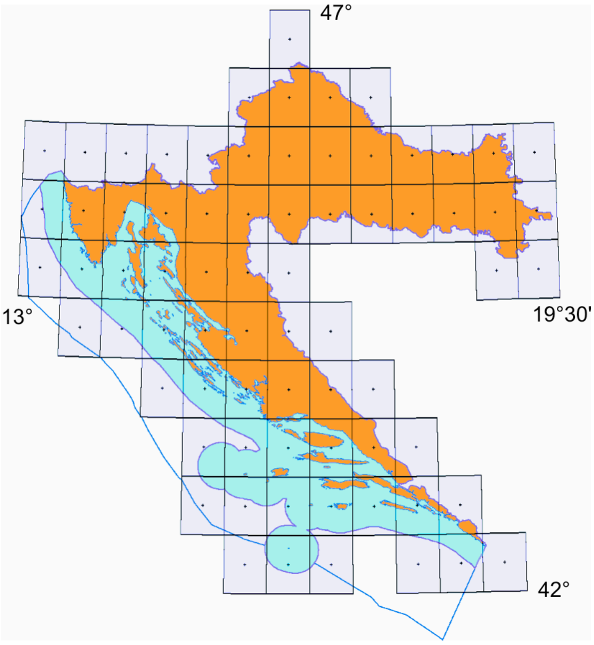

Figure 2. Approximation of the territory of Croatia with a grid of ellipsoidal quadrangles of size 30'

For each quadrangle, the linear scale in the middle point B_i, L_i is defined by the equation:

$$c_i = \frac{2\alpha_0 R_0 \cos\varphi_i}{k_i N_i \cos B_i}$$

where

$$\tan\left(\frac{\pi}{4} + \frac{\varphi_i}{2}\right) = \frac{1}{K_0} \tan^{\alpha_0}\left(\frac{\pi}{4} + \frac{B_i}{2}\right)\left(\frac{1 - e\sin B_i}{1 + e\sin B_i}\right)^{\frac{\alpha_0 e}{2}}$$

$$\lambda_i = \alpha_0 L_i$$

$$k_i = 1 + \sin\varphi_0 \sin\varphi_i + \cos\varphi_0 \cos\varphi_i \cos(\lambda_i - \lambda_0)$$

$$N_i = \frac{a}{\left(1 - e^2 \sin^2 B_i\right)^{\frac{1}{2}}}.$$

The area ΔA_i of an ellipsoidal quadrangle with the middle point B_i, L_i can be computed [8]:

$$\Delta A_i = \frac{b^2 \Delta L}{2}\left(\frac{\sin B}{1 - e^2 \sin^2 B} + \ln\left(\frac{1 + e\sin B}{1 - e\sin B}\right)^{\frac{1}{2e}}\right)\Bigg|_{B_i - \frac{\Delta B}{2}}^{B_i + \frac{\Delta B}{2}}$$

where b is the minor semi-axis of the ellipsoid. GRS80 was the ellipsoid used in computations [10].

The criterion (3) was minimized for parameters B_0, L_0 to precision of 1' (~2km on Earth). The solution was found using a program written in C++ using a brute-force approach. For quadrangles of size 1°, 30' and 10' the solution was also found using *Mathematica 5.1* and its function *NMinimize* which tries to find a global minimum of a given function [15]. It was used to double check the calculation. Table 1 gives the results. The search range for value B_0 was [42°, 47°] and for L_0 [13°, 20°]. The step was 1'. For the last two grids (2' and 1'), the search range was narrowed to [44°, 46°] for B_0 and to [15°, 17°] for L_0.

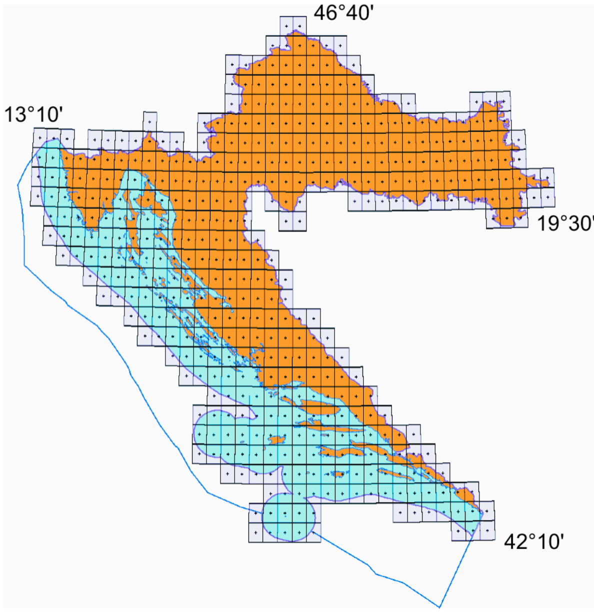

Figure 3. Approximation of the territory of Croatia with a grid of ellipsoidal quadrangles of size 10'

Table 1. Optimal parameters B_0 and L_0 for Croatia for different sizes of approximating quadrangles.

size of q.	E	B_0	L_0
1°	$2.6928 \cdot 10^{-4}$	44°30'	16°33'
30'	$2.5255 \cdot 10^{-4}$	44°25'	16°22'
10'	$2.1114 \cdot 10^{-4}$	44°28'	16°22'
5'	$2.0250 \cdot 10^{-4}$	44°28'	16°23'
2'	$1.9699 \cdot 10^{-4}$	44°28'	16°21'
1'	$1.9428 \cdot 10^{-4}$	44°28'	16°21'

As final parameters using this method we can take: $B_0 = 44°28'$, $L_0 = 16°21'$ (Figure 4). The constants α_0, K_0, R_0, φ_0 and λ_0 can be calculated by using (2). The numerical values with 10 significant digits are:

$$\alpha_0 = 1.000873713$$
$$K_0 = 0.9972633826$$
$$\varphi_0 = 44°25'03.3367''$$
$$\lambda_0 = 16°21'51.4267''$$
$$R_0 = 6377702.298\,.$$

2.2 Calculation of parameters – the second approach

For the purpose of more optimal distribution of map projection distortions, a constant scale factor m_0 is usually applied to coordinates x, y in the projection plane. By using such a scale factor m_0, the formulas for plane coordinates become:

$$x = m_0 \frac{2R_0}{k}\left(\sin\varphi\cos\varphi_0 - \sin\varphi_0\cos\varphi\cos(\lambda-\lambda_0)\right)$$
$$y = m_0 \frac{2R_0}{k}\sin(\lambda-\lambda_0)\cos\varphi \qquad (7)$$

with

$k = 1 + \sin\varphi_0\sin\varphi + \cos\varphi_0\cos\varphi\cos(\lambda-\lambda_0)$.

The linear scale factor is now obviously:

$$c = \frac{2m_0\alpha_0 R_0\cos\varphi}{kN\cos B}\,. \qquad (8)$$

This time, by using the same procedure as in the first attempt, we have to find three parameters B_0, L_0 and m_0 in order to minimize the Airy/Jordan criterion.

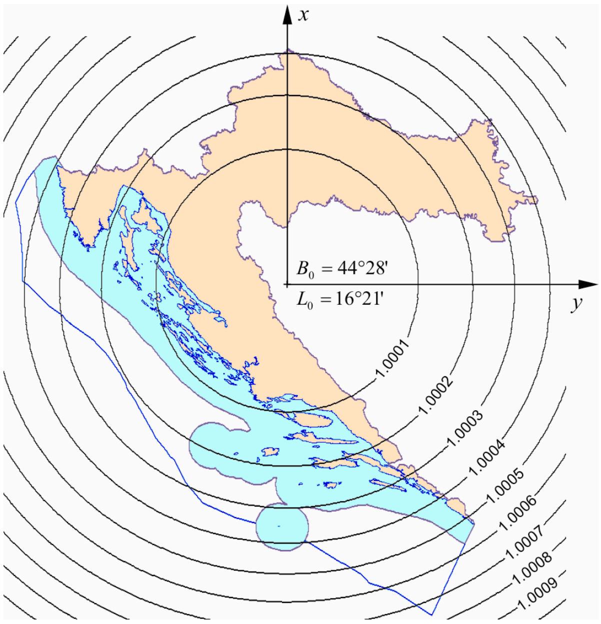

Figure 4. Lines of constant scale in the first variant of stereographic projection of Croatia with parameters $B_0 = 44°28'$, $L_0 = 16°21'$

For quadrangles of size 1°, 30' the solution was found using the function *NMinimize* in *Mathematica 5.1* to double check the calculation. Table 2 gives the results. The search range for value B_0 was [42°, 47°], for L_0 [13°, 20°] and for m_0 [0.99, 1]. The step was 1' for B_0 and L_0, and the step for m_0 was 0.00001. The search range was narrowed to [44.2°, 44.6°] for B_0, to [16.4°, 16.8°] for L_0 and to [0.999, 1] for m_0 for last two grids (2' and 1').

Table 2. Optimal parameters B_0, L_0 and m_0 for Croatia for different sizes of approximating quadrangles.

size of q.	E	B_0	L_0	m_0
1°	$1.3099 \cdot 10^{-4}$	44°29'	16°38'	0.99976
30'	$1.3574 \cdot 10^{-4}$	44°23'	16°31'	0.99979
10'	$1.1094 \cdot 10^{-4}$	44°25'	16°34'	0.99982
5'	$1.0486 \cdot 10^{-4}$	44°25'	16°36'	0.99982
2'	$1.0230 \cdot 10^{-4}$	44°25'	16°34'	0.99983
1'	$1.0079 \cdot 10^{-4}$	44°25'	16°34'	0.99983

The final parameters are: $B_0 = 44°25'$, $L_0 = 16°34'$, $m_0 = 0.99983$ (Figure 5). The constants α_0, K_0, R_0, φ_0 and λ_0 can be calculated by using (2). The numerical values with 10 significant digits are:

$$\alpha_0 = 1.000876707$$
$$K_0 = 0.9972700482$$
$$\varphi_0 = 44°22'03.0409''$$
$$\lambda_0 = 16°34'52.2868''$$
$$R_0 = 6377664.924\,.$$

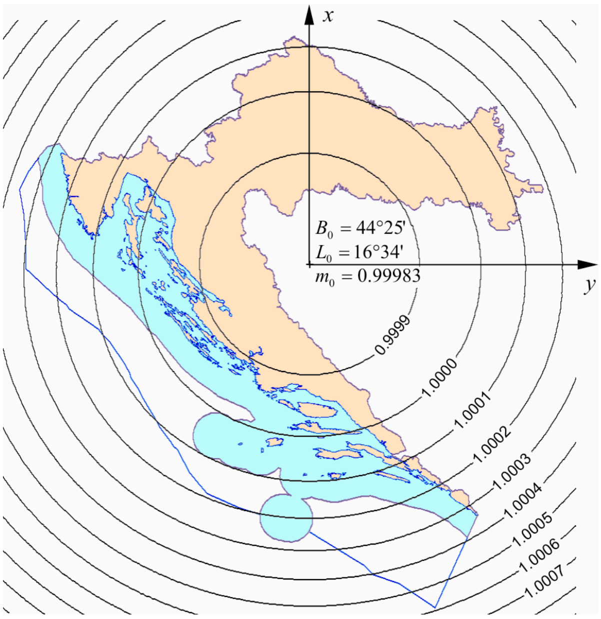

Figure 5. Lines of constant scale in the second variant of stereographic projection of Croatia with parameters $B_0 = 44°25'$, $L_0 = 16°34'$ and $m_0 = 0.99983$

2.3 Calculation of parameters – the third approach

The third approach was to find values of five parameters B_0, L_0, α, K and R for which minimum of (6) is obtained. It means that the values for α, K and R will not be calculated from (2) as α_0, K_0 and R_0, but the values will be found numerically using (6).

For each quadrangle, the linear scale in the middle point B_i, L_i is defined by the equation:

$$c_i = \frac{2\alpha R \cos\varphi_i}{k_i N_i \cos B_i}$$

where

$$\tan\left(\frac{\pi}{4}+\frac{\varphi_i}{2}\right) = \frac{1}{K}\tan^{\alpha}\left(\frac{\pi}{4}+\frac{B_i}{2}\right)\left(\frac{1-e\sin B_i}{1+e\sin B_i}\right)^{\frac{\alpha e}{2}}$$

$$\lambda_i = \alpha L_i$$

$$k_i = 1+\sin\varphi_0\sin\varphi_i+\cos\varphi_0\cos\varphi_i\cos(\lambda_i-\lambda_0)$$

$$N_i = \frac{a}{\left(1-e^2\sin^2 B_i\right)^{\frac{1}{2}}}$$

$$\tan\left(\frac{\pi}{4}+\frac{\varphi_0}{2}\right) = \frac{1}{K}\tan^{\alpha}\left(\frac{\pi}{4}+\frac{B_0}{2}\right)\left(\frac{1-e\sin B_0}{1+e\sin B_0}\right)^{\frac{\alpha e}{2}}$$

$$\lambda_0 = \alpha L_0\,.$$

This time we have to find the minimum of the function (6) with five unknown parameters. In previous calculations the function *NMinimze* in *Mathematica 5.1* gave same results when different methods (option *Method*) were used or bigger accuracy required (option *AccuracyGoal*). For third approach, function *NMinimze* in *Mathematica 5.1* gave different results on 1° grid using different methods for finding the minimum (Table 3). Option *AccuracyGoal* was set to 24, range for B_0 was [42°, 46°], for L_0 range was [16°, 17°], for α and K range was [0.9, 1.1] and for R range was [6360000, 6380000]. All solutions were found below default maximum number of iterations (option *MaxIterations*) which is 100. The results in Table 3 show that determination of parameters using this approach is more difficult and deserves further investigation. Finding solution using brute-force approach in case of five unknown parameters is extremely time consuming.

Table 3. Parameters B_0, L_0, α, K and R calculated with *NMinimze* in *Mathematica 5.1* using different methods for quadrangles of size 1°.

	1	2	3	4
E	$1.30918 \cdot 10^{-4}$	$1.30918 \cdot 10^{-4}$	$1.30918 \cdot 10^{-4}$	$1.30918 \cdot 10^{-4}$
B_0	45°01'	44°54'	44°48'	44°57'
L_0	16°38'	16°38'	16°38'	16°38'
α	0.999168	0.999168	0.999168	0.999168
K	1.000095	0.998667	0.997380	0.999272
R	6367731	6374142	6379926	6371427

1 – *SimulatedAnnealing* method
2 – *NelderMead* method
3 – *DifferentialEvolution* method
4 – *RandomSearch* method

CONCLUSIONS

In general, the stereographic projection of an ellipsoid is not defined in a unique way. One can find different approaches in literature. The Kavrayskiy's research [6] on stereographic projections of ellipsoid is interesting. His definition states that any generalisation which gives known stereographic projection of sphere in special case, when ellipsoid is replaced by sphere ($e = 0$), can be considered as a stereographic projection of ellipsoid.

The approach defined by C. F. Gauss is especially convenient because of simplicity and avoidance of power series [6]. We applied this approach for mapping the territory of the Republic of Croatia. Various criteria can be applied to find the parameters of mapping. We have presented two variants using the Airy/Jordan criterion. The first variant has two parameters, and its optimal values are $B_0 = 44°28'$ and $L_0 = 16°21'$. The second variant has three parameters and its optimal values are $B_0 = 44°25', L_0 = 16°34'$ and $m_0 = 0.99983$. We propose further investigation for finding optimal parameters in variant with five unknown parameters.

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