HEIGHT TRANSFORMATION MODELS FROM ELLIPSOIDAL INTO THE NORMAL ORTHOMETRIC HEIGHT SYSTEM FOR THE TERRITORY OF THE CITY OF ZAGREB

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ABSTRACT

The paper presents the testing of the possibility of determining the heights of GPS points in the homogeneous field in the new Croatian Height Reference System (HVRS71) by using the method of height transformation. The testing was made in the area of Zagreb. As part of the field works, normal orthometric heights of 27 GPS points were determined according to the new height system, by transferring the benchmark heights using the geometric levelling method, thus obtaining GPS/levelling points of known ellipsoidal and normal orthometric heights. The GPS/levelling points served as the basis for determining the transformation models that enabled the computation of normal orthometric heights from ellipsoidal heights of any GPS point in the observed area. The empirical data used for modelling were reduced undulation dN values of GPS/levelling points. As part of the dN modelling with parametric functions, the approximation surfaces were obtained on the basis of three polynomials: FN310, FN312 and FN318. The transformation models were also tested using non-parametric Watson and Loess algorithms. The FN318 and Loess models yielded the best results.

Key words: GPS/levelling points, transformation, Croatian Height Reference System 1971 (HVRS71), normal orthometric height, ellipsoidal height, undulation

1. INTRODUCTION

The use of heights, such as orthometric or normal orthometric heights, that are connected to the Earth's gravity field, is important in many fields ranging from geodesy, civil engineering, geophysics, oceanography, etc. Traditionally, these heights are determined by combining geometric levelling and gravity acceleration measurements. However, in the last twenty years, the wide and propagating use of GPS in height determination has incited the need to define a more accurate geoid or quasigeoid model in order to make the determination of orthometric, normal or normal orthometric heights possible with sufficient accuracy in a simple and cost-effective way (*Heiskanen and Moritz, 1996*; *Torge, 1989, 2001*).

Namely, the GPS ellipsoidal heights referring to a reference ellipsoid have a more theoretical than practical significance. They can be easily transformed into altitudes using a simple formula Eq.(1), in order to be compatible with the local height datum. The problem arises when transforming these heights into one of the currently used height reference systems, because it is necessary to know the reference surface (geoid, quasigeoid, etc.) whose accuracy is in accordance with the accuracy of GPS ellipsoidal heights and altitudes, i.e. should not be bigger than a few centimetres.

The basic formula for transforming ellipsoidal heights (h) into a height reference system (HRS) reads (*Dinter et al., 1997*)

$$H = h - N \,. \tag{1}$$

Fig. 1 presents the relation between the ellipsoidal height h and the orthometric (normal, normal orthometric) height H. Depending on the reference surface of the selected height system (HRS) the previous expression reads as follows:

$$H = h - N_{geoid} \,, \tag{2}$$

$$H^N = h - N_{quasigeoid} \,, \tag{3}$$

$$H^\gamma = h - N_\gamma \,, \tag{4}$$

where in the fist case N_{geoid} (geoid undulation) refers to geoid as a reference surface, and the resulting height is defined as orthometric height. In the second case $N_{quasigeoid}$ (height anomaly) refers to quasigeoid, and the H^N height is called normal height. In the third case H^γ is calculated using only the normal value of the gravity field acceleration of the reference γ ellipsoid along the levelling path, and it is called normal orthometric height.

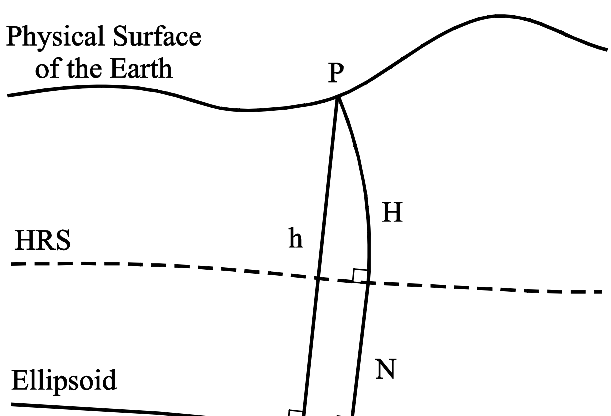

Fig. 1. Relation between ellipsoidal height h, height H and height anomaly N.

Accordingly, N_γ is called normal orthometric height anomaly. All three types of N's can be commonly called "height anomaly" and, as shown in Fig. 1, the difference between ellipsoidal height h and the plumb line can be neglected. Namely, the differences in their real values are always smaller than 1 mm (*Dinter et al., 1997*).

This problem of transformation can be solved if the following criteria are met (*Dinter et al., 1997*):

- the height anomaly N, referring to the height reference system (HRS) where the height H is to be found, is known;
- the geoid model of the quasigeoid is compatible with the H height or measured height differences ΔH (N's refer to the same type of heights);
- the height anomalies refer to the same ellipsoid and datum as well as ellipsoidal heights h;
- there is no systematic influence of height anomalies N.

Hence, in order to determine orthometric, normal or normal orthometric heights by using GPS, it is necessary to determine the height anomalies with sufficient accuracy. However, height anomalies N do not exist for some local areas or their density is insufficient. If there is a gravimetric geoid model for a local area, the geoid accuracy can be increased by including GPS/levelling points whose ellipsoidal height h and orthometric (normal, normal orthometric) height H are known. In smaller local areas, (a few dozens kilometres) it is possible to determine the mathematical model by means of a certain number of GPS/levelling points that can be used for the interpolation of GPS points whereby only the ellipsoidal height is known. An adequate model will enable the determination of the orthometric (normal, normal orthometric) height of GPS points with a certain level of accuracy. The size that is modelled, i.e. the one that is included in the model, is a reduced value of the dN undulation. The dN is obtained by subtracting the undulation mean value from undulation of individual GPS/levelling points.

$$H = h - \left[N_{average} + dN \right] . \tag{5}$$

There are a few modelling methods for height anomalies (undulation, geoid heights) that can be used for the transformation of ellipsoidal heights into orthometric (*Zhong, 1997*), but one of the simplest methods of determining the surface is based on the approximation (interpolation) using a polynomial. Here, two types of models based upon the polynomial approximation will briefly be presented, although in essentially different concepts. One of them is the model of transformation using the parameter function. It will explain the polynomial interpolation model. The other transformation model is based on modelling the surface by means of special algorithms containing the lower order polynomials. It should be pointed out that the accuracy of every single model depends greatly on the accuracy of the input data but also on the adequately selected mathematical function used for modelling the height anomalies dN. Since both models have been tested for the City of Zagreb, the text below presents the comparison and analysis of normal orthometric heights of GPS/levelling points belonging to this test area (*Bašić et al., 1999*; *Bašić, 2001*; *Čolić, 1998*; *Čolić et al., unpublished results*; *Švehla, 1997*).

2. THE RESULT ANALYSIS OF INDIVIDUAL TRANSFORMATION MODELS

The basic input data used for calculating and analyzing the individual transformation model are given in Table 1 (*Gucek, 2005*). The basic information about each GPS/levelling point is given in the same table. i.e. the name of the area, GPS point number, positional coordinates of GPS points given in the Gauss Krüger projection, and the height component which is the reduced undulation value as related to the new datum definition in HVRS71 (*Official Gazette, 2001, 2004*; *Feil et al., 2003*).

The calculations and analysis of every individual model of transforming the ellipsoidal heights of GPS points into normal orthometric heights in the new HVRS71 height system were made on the basis of the reduced undulation values for 27 discrete GPS/levelling points of the network of the City of Zagreb shown in Fig. 2.

Altogether five models have been used for determining the reduced undulation values *dN* tested: three parameter models numbered with Taylor's polynomials *FN*310, *FN*312 and *FN*318, and two non-parametric models called Watson and Loess approximations.

Table 1. Basic data about GPS/levelling points.

No.	Area	GPS	y_{GK} [m]	x_{GK} [m]	dN_{HVRS71} [m]
1	Sesvetski Kraljevec	1018	5592337	5074305	–0.2747
2	Horvati	1037	5564401	5063714	–0.0131
3	Kašina	1237	5587433	5085433	0.1965
4	Sljeme	1331	5574396	5084270	0.5092
5	Belovar	1476	5591441	5082794	–0.0448
6	Miroševec	1781	5580983	5080498	0.0895
7	Štefanovec	1840	5579836	5080065	0.2582
8	Markovo polje	1982	5586727	5079256	–0.1197
9	Sesvete	2573	5586722	5076477	–0.0997
10	Podsused	2795	5565166	5075689	0.1499
11	Podsused	3010	5565903	5074928	0.1269
12	Centar	3040	5575731	5074804	0.0540
13	Vrapče	3080	5570775	5074723	0.0702
14	Borongaj	3215	5579705	5074281	–0.0247
15	Ivanja Reka	3453	5586672	5073397	–0.1282
16	Ježdovec	3961	5565503	5070830	0.0580
17	Petruševac	4026	5581858	5070325	–0.1264
18	Otok	4204	5577440	5069073	–0.1201
19	Hrašće	4497	5579049	5066108	–0.1380
20	M. Mlaka-Vodovod	4501	5576109	5066090	–0.2257
21	Brezovica	4578	5571386	5065116	–0.0483
22	Beduri	4750	5560552	5062923	0.0458
23	Lipnica	4862	5573975	5060836	–0.0560
24	Vidalin	4870	5561024	5060710	0.0017
25	Kupinečki Kraljevec	4978	5568344	5058421	–0.0226
26	Donji Dragonožec	4992	5573791	5058053	–0.0566
27	Šestine	5081	5574152	5079201	0.2631

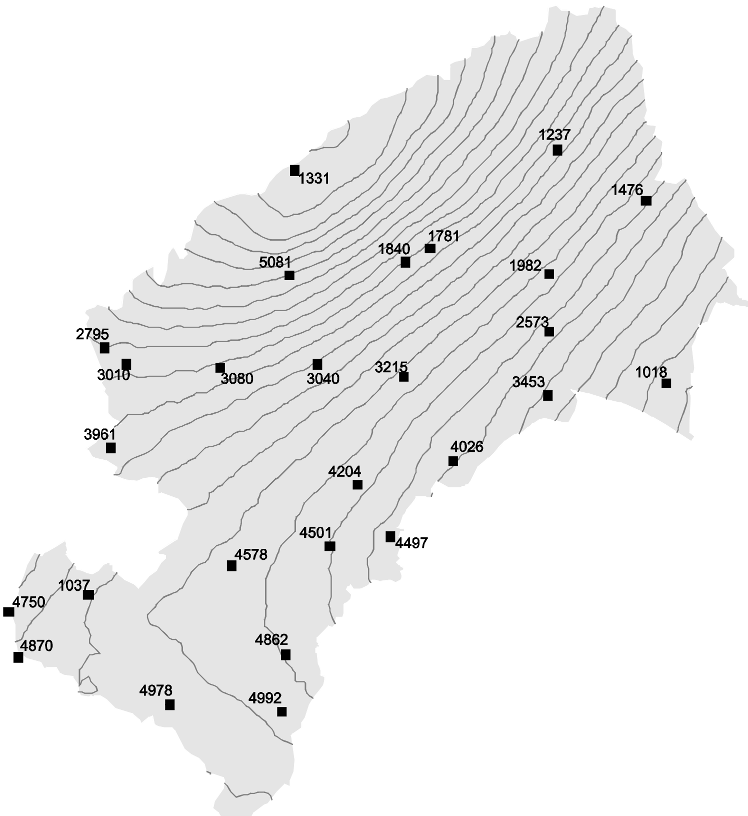

Fig. 2. GPS/levelling points in the City of Zagreb.

Each model made it possible to calculate a reduced undulation (*dN*) value on the basis of positional coordinates of GPS points in the Gauss Krüger projection (y_{GK}, x_{GK}) , as well as to determine the normal orthometric height (*H*) is defined in the new height system of the Republic of Croatia, HVRS71 shown in Eq.(6) by adding the valued *dN* to the mean value of undulation (*N*) and by subtracting from ellipsoidal height in average the GRS80 system (*h*).

$$H^i_{\mathrm{HVRS71}} = h^i_{\mathrm{GRS80}} - \left[N_{average} + dN^i \right], \qquad i = 1, \ldots n . \tag{6}$$

The selection of the best model describing the empirical data is generally not defined by a unique statistical procedure that can answer this question exactly. One can thereby select a few model accuracy estimation criteria that can be used. These are standard deviation σ, coefficient of determination R^2, adjusted coefficient of determination R^2_{adj} , etc. The above-mentioned accuracy estimation measures and statistical projections make it possible to select the most adequate data model, but it is our knowledge and professional competence regarding the problems, as well as the analytical approach that should at the end be decisive in selecting the model that can best describe the empirical data.

The following are a few important factors that influence the selection of the model as well as the accuracy of the model itself (*Draper and Smith, 1998*):

- distribution and number of discrete GPS/levelling points that have to be selected in such a way as to cover the entire modelled area, equally positioned in order to define, in particular, the irregularity of the terrain, i.e. the topology;
- accuracy of ellipsoidal and normal orthometric heights of the GPS/levelling points;
- topology of the modelled area;
- selection of the modelling method (different areas require different methods).

All calculations have been made by the TableCurve3D version 4.0 application package, a specialized scientific software used in various fields of the engineering, medical, natural and other sciences (*Analytical graphics package TableCurve3D, http://www.systat.com/products/TableCurve3D*).

2.1. *dN* Modelling with Parametric Functions

One of the most frequent and most efficient mathematical model applied in modelling a geoid surface in a smaller area is the surface approximation using polynomial, i.e. polynomial interpolation. This method is also very often used to define long-wave geoid surface trends. The degree of the bivariant polynomial should be determined depending on the size and topography of the area. The basic principle of *dN* modelling with parameter functions is the selection of parameters that are directly connected with the features of the data modelled. The models of parameter functions are sensitive to the number of parameters and to the connection, i.e. correlation of parameters.

Provided that there are n well distributed GPS/levelling points in a selected area, with known GPS ellipsoidal height h and the orthometric (normal, normal orthometric) height H determined by levelling, it is recommended to know a geoid model as well. Eq.(7) represents a general form of the function $N(x, y)$ that makes it possible to determine the height anomaly (geoid height, undulation) of a GPS point, having the known ellipsoidal height h in this area and known positional coordinates y and x (*Zhong, 1997*)

$$N(y,x) = N^0(y,x) + dN_{MODEL}(y,x) , \tag{7}$$

where

$$dN_{MODEL}(y,x) = \sum_{i=0}^{m} \sum_{j=0}^{m-i} a_{ij} y^i x^j , \quad i, j = 0,\ldots,m .$$

$N(y, x)$ represents height anomaly (geoid height, undulation) of the point with known coordinates (y, x) and ellipsoidal height h, $N^0(y, x)$ is the height determined for some other geoid model, and $dN_{MODEL}(y, x)$ presents a reduced value of the height anomaly mathematically defined by a polynomial. The parameters of the model a_{ij}, i.e. the coefficients of m-degree polynomial $(i, j = 0, \ldots, m)$ are not known and should be determined by means of adjustment.

In order to apply the method of least squares in this system of n equations with m unknown coefficients, nonlinear functions should be developed into linear by using the second-order Taylor series. Hence, for a redundant number of GPS/levelling points it is possible to determine the following system of equations:

$$\boldsymbol{v} = \mathbf{A}\boldsymbol{x} - \boldsymbol{l} , \tag{8}$$

where

$$\boldsymbol{l} = \begin{pmatrix} h_1 - H_1 - N^0 \\ \vdots \\ h_n - H_n - N^0 \end{pmatrix} = \begin{pmatrix} l_1 \\ \vdots \\ l_n \end{pmatrix} . \tag{9}$$

Vector $\boldsymbol{v}$ represents the residuals (corrections), i.e. deviations from the model, $\mathbf{A}$ is an $n \times m$ coefficient matrix of the equation system, and vector $\boldsymbol{x}$ contains unknown parameters of the model. Matrix $\mathbf{A}$ contains the coefficients obtained through the derivation of the $dN_{MODEL}(x, y)$ function with unknown a_{ij} coefficients. Vector $\boldsymbol{l}$ is calculated by means of the known GPS ellipsoidal heights of discrete GPS/levelling points h_i, orthometric (normal, normal orthometric) heights of GPS points H_i and the known approximate (mean) value of height anomalies (geoid height, undulations) N^0. The least squares adjustment is made by minimizing the $\boldsymbol{v}^T \mathbf{P} \boldsymbol{v} \rightarrow min$ objective function. In case of equal accuracy of the input GPS points data, $\mathbf{P}$ presents a unit matrix $\mathbf{P} = \mathbf{I}$. By using normal equations, one obtains the solution of unknown parameter sizes contained in vector $\bar{\boldsymbol{x}}$

$$\bar{\boldsymbol{x}} = \left(\mathbf{A}^T \mathbf{P} \mathbf{A}\right)^{-1} \mathbf{A}^T \mathbf{P} \boldsymbol{l} . \tag{10}$$

The variance and covariance matrix of the unknown polynomial parameters reads as follows

$$\mathbf{V}_{xx} = \boldsymbol{\sigma}^2 \mathbf{Q}_{xx} = \boldsymbol{\sigma}^2 \left(\mathbf{A}^T \mathbf{P} \mathbf{A}\right)^{-1} , \tag{11}$$

where the variance $\boldsymbol{\sigma}^2$ is expressed by

$$\boldsymbol{\sigma}^2 = \frac{\boldsymbol{v}^T \mathbf{P} \boldsymbol{v}}{n - u} . \tag{12}$$

The positive value of the variance square root presents the standard σ deviation, as one of the criteria of the approximation accuracy estimation, i.e. one of the indicators of how the model is adjusted to the data. Hence, depending on the equation system solution, the obtained mathematical function defines exactly and unanimously the change pattern of dN in the observed area. Since the approximation function is continuous and limitless function, it enables the analytical determination of a dependent and unknown dN on the basis of independent and known x and y coordinates.

In modelling the geoid surface by means of the polynomial approximation method, one should take into consideration the accuracy of the input data by adding a certain weight to each of them. Thus, the points of smaller weight will, at the same time, have less influence on the model. In the process of adjustment, the negative influence of less accurate data (low measurement accuracy, gravity acceleration anomalies in the local area, etc.) will be reduced to the possible abnormal model behaviour (*Zhong, 1997*). Furthermore, one of the ways to preserve approximation accuracy is the number and selection of polynomial parameters that influence directly the form and composition of the **A** matrix coefficient. In order to achieve better accuracies of unknown parameters of the polynomial and a more accurate and less "biased" approximation, i.e. smaller systematic influence, it is recommendable to remove insignificant parameters from the model. This all leads to the conclusion that a good model should contain a small number of significant parameters (*Draper and Smith, 1998*). Namely, it is recommended to remove insignificant parameters in a model where only one function for the modelling of the whole surface and where all discrete points are used in modelling, in order to make approximation function as simple as possible (*Soycan and Soycan, 2003*). However, it should be taken into consideration that the removal of insignificant parameters from the model will not always yield satisfactory solutions. A good selection of function parameters, i.e. the removal of insignificant parameters, and the acceptance of the significant ones require certain predefined criteria, i.e. the testing of all parameters of the selected model. In practice, the F-test (Fisher test) is used to compare and judge the significance of parameters.

Transformation Models Using Parametric Functions

In the TableCurve3D software package, the criteria are set for the selection of the best approximation curves based on the standard σ deviation (Fit standard error - Fit Std)

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} p_i \left(dN^i_{model} - dN^i_{observation} \right)^2}{f}} , \tag{13}$$

where f is the degree of freedom, and p_i are the weight coefficients.

Taking into consideration all 27 GPS/levelling points and based on the Surface-Fit custom Equations set function, the three polynomials (*FN*310, *FN*312 and *FN*318) selected will be further considered and analyzed in Eq.(14)

$$\begin{aligned} FN310:&\ z = a + bx + cy + dx^2 + ey^2 + fxy + gx^3 + hy^3 + ixy^2 + jx^2 y , \\ FN312:&\ z = a + \frac{b}{x} + cy + \frac{d}{x^2} + ey^2 + \frac{fy}{x} + \frac{g}{x^3} + hy^3 + \frac{iy^2}{x} + \frac{jy}{x^2} , \\ FN318:&\ z = a + \frac{b}{x} + \frac{c}{y} + \frac{d}{x^2} + \frac{e}{y^2} + \frac{f}{xy} + \frac{g}{x^3} + \frac{h}{y^3} + \frac{i}{xy^2} + \frac{j}{x^2 y} . \end{aligned} \tag{14}$$

Each equation is the function of the known positional x and y coordinates, on the basis of which the unknown z component is calculated, presenting in this problem the reduced undulation value of GPS point dN_i. The unknown parameters, i.e. coefficients of each function, are calculated with the adjustment on the basis of 27 GPS/levelling points using the Gauss method of least squares (see Section 2.1.).

For each model, Fit Std (Fit standard deviation) indicates only the internal model accuracy. However, to compute external accuracy it is necessary to calculate the value of the height of the additional set of the control GPS points on the basis of the model, and to compare them with the data obtained by field measurements. Thus, the deviations obtained represent true model errors provided that the input data are values without errors. Since no additional field measurements have been made, the same points used for modelling are used as control points. Such a procedure of using the data that have been applied in the processing and in the checking procedure is known as "cross validation method" (*Judging Model Quality by Residuals, http://www-2.cs.cmu.edu/~schneide/tut5/node42.html*). The calculated parameters are slightly changed, thus obtaining surface approximations that can be considered as identical. The control point that has not been used in the surface approximation has served in calculating the deviation as the "true error" of the model. The deviations of control points have also been compared with the standard deviation, i.e. with internal accuracy of the model itself. The final estimation of an individual model has been made on the basis of the percentage of points that have met the 3σ criteria.

Further in the text the basic data will be presented, making it is possible to analyze in more details only the *FN*310 and *FN*312 models, as well as *FN*318. Model *FN*318 has been taken into consideration in spite of the fact that its application without point 1018 is not possible.

Accuracy Estimation of Individual Parametric Transformation Models

In order to characterize each height transformation model as good as possible, it is necessary to first explain which external model accuracy is realistic to expect. The expected model accuracy can not be better than the accuracy of GPS/levelling points, i.e. the input data used in transformation models. The model accuracy will surely depend on the benchmark accuracy, as well as on the accuracy of points from which the height has been transformed as well as the accuracy of the GPS point's heights themselves.

If we take into consideration the fact that benchmarks are fixed quantities without errors and if we neglect the correlation, the accuracy of normal orthometric heights of GPS points is expressed by the standard deviation of unknowns in the adjustment of every single levelling side or levelling figure (*Gucek, 2005*). Standard deviation of normal orthometric heights of GPS points, as determined by only one benchmark, has the value smaller than 1 mm, and standard deviations of normal orthometric height of GPS points, as determined by two benchmarks in the levelling figure, are larger and are between 0.5 mm and 7.7 mm.

Table 2 presents the accuracies of node bench marks and GPS points contained in the previous calculations, with only their extreme and mean values because one wants to have simple concise overview of the input data accuracy. However, one should point out once again, that these numerical values do not present standard deviations of the height of individual benchmarks or individual GPS points. By applying the law of error propagation

Table 2. Input data accuracy estimation.

Node Benchmark Accuracy $\sigma_{Benchmark}^{H^\gamma}$ [cm]		GPS Height Accuracy $\sigma_{GPS}^{H^\gamma} = \sqrt{\sigma_{Benchmark}^2 + \sigma_{GPS}^2}$ [cm]		
			2 cm	3 cm
Min	0.04	Min	2.00	3.00
Max	5.27	Max	5.64	6.07
Min Avg	0.34	Min Avg	2.03	3.02
Max Avg	1.41	Max Avg	2.44	3.31

one could calculate standard deviations of the input data combined with the minimum, maximum, average minimum and average maximum standard deviations of node benchmarks of all orders in the levelling networks and with standard deviations of the ellipsoidal height of GPS points in the homogeneous field being from 2 to 3 cm (*Niemeier, 2002*; *Feil, 1989*).

The accuracy of normal orthometric height of GPS points is between 2.00 and 6.07 cm. In this respect, Table 3 shows the results that give the number and percentage of the total number of points in certain accuracy categories (< 1 cm, < 2 cm, < 3 cm, < 4 cm, < 5 cm, < 6 cm). For each model about 40% of control GPS points have the absolute value of deviation smaller than 3 cm, and 45% to 60% the absolute value of deviation smaller than 5 cm. In the *FN*312 model, almost 75% of points have the absolute value of residual smaller than 6 cm.

The text below will present the estimation of the internal and external model transformation accuracy using the *FN*310, *FN*312 and *FN*318 polynomials. In the *FN*318 model, the GPS point 1018-Sesvetski Kraljevec has been excluded from the accuracy estimation calculation due to before-mentioned reasons.

Every residual consists of two kinds of influences: random and gross errors (*Kavouras, 1982*). The accuracy of the input data is represented by the random component while the remaining part is the systematic influence of the mathematical model itself.

Table 4 presents the minimum and maximum absolute value of residuals, square deviation sum, variances and the standard deviation in centimetres, for each individual model.

Table 3. Percentage of the number of points having the accuracy of between 0 and 6 cm.

σ	*FN*310		*FN*312		*FN*318	
	Number of Points	%	Number of Points	%	Number of Points	%
< 1 cm	4	14.5	3	11.1	5	19.2
< 2 cm	7	25.9	10	37.0	7	26.9
< 3 cm	10	37.0	11	40.7	10	38.5
< 4 cm	12	44.4	13	48.1	11	42.3
< 5 cm	12	44.4	16	59.3	16	61.5
< 6 cm	16	59.3	20	74.1	17	65.4

Table 4. Internal model accuracy estimation.

	*FN*310 [cm]	*FN*312 [cm]	*FN*318 [cm]
MinAbs(r) (Number of GPS Point)	0.30 (3040)	0.05 (4497)	0.11 (1331)
MaxAbs(r) (Number of GPS Point)	9.28 (4501)	11.58 (4501)	10.51 (4501)
Σr^2	4.42	4.44	4.23
σ^2	26.01	26.11	24.90
σ	5.10	5.11	4.99

Table 5. External model accuracy estimation.

	*FN*310 [cm]	*FN*312 [cm]	*FN*318 [cm]
MinAbs(r) (Number of GPS Point)	0.03 (3040)	0.28 (1037)	0.00 (1037)
MaxAbs(r) (Number of GPS Point)	26.99 (1081)	13.77 (4501)	12.70 (4501)
$\Sigma\sigma^2$	7.76	8.69	6.64
Average σ^2	5.36	5.67	5.05

From the above stated one can conclude that all models have approximately the same accuracy, although the accuracy of the *FN*318 model is a few millimetres better.

The detail processing, i.e. the presentation of the external model accuracy, can lead to clearer conclusions. As it has been said previously, the estimation of the external model accuracy has been made on the basis of true errors of control GPS points, i.e. for each of the 27 points. Thereby, the *FN*310, *FN*312 and *FN*318 models have been clearly observed, but in 27 (and the *FN*318 model in 26) different combinations (function coefficients have been slightly changed in each combination). Since there are different functions involved here (with various coefficients), although from the same group of polynomials, the total variance and standard deviation have not been computed, but the sum of the $\Sigma\sigma^2$ variance in order to calculate the average value of σ^2 . Table 5 presents all the data, and there are also the data expressed for the minimum and maximum absolute values of residuals.

It is very important to point out that the residuals of control GPS points, calculated from models, present true errors, i.e. deviations of the individual reduced undulation values for control GPS points (computed from the remaining 26 points) from the measured value, and that they are the true accuracy indicators. Table 5 shows the largest differences between minimum and maximum values of almost 27 cm for the *FN*310 model, and the smallest for the *FN*318 model of 12.70 cm whereby GPS point 1018 has not been taken into account. It also points out the large extreme values of residuals obtained by extrapolating all the points located at the edges of the areas or for those points located in the areas of sudden elevation or slope of the terrain. Therefore, the GPS point's 1018-Sesvetski Kraljevec in the *FN*310 model and 4501-Vodovod_M.Mlaka in the *FN*312 and *FN*318 models obtain maximum residual values. Table 5 shows that the points that crop out for the *FN*310 and *FN*318 models have minimum and maximum residual values (GPS points 1037-Horvati and 4501-Vodovod_M.Mlaka), which is a logical phenomenon because the approximation functions are the same up to the third polynomial degree. It should still be mentioned that in the *FN*318 model, the GPS point 1018-Sesvetski Kraljevec has maximum residual value and runs up to $r_{1018} = \pm\infty$. Namely, after removing this point from the calculation it is not possible to calculate the *FN*318 approximation function, and hence the residual itself.

Table 6 shows the external accuracy estimation for the model computed on the basis of 26 (25) GPS/levelling points. Each single point has later on served as the control point for these models.

It can be seen from the table that almost 75% of the residuals of control GPS points obtain the absolute values smaller than 2σ, hence, smaller than the average 10 cm. It should also be pointed out that in the *FN*312 and *FN*318 models about 60% of points have the residual values smaller than 1σ, i.e. smaller than 5 cm. One speaks about external accuracy, hence, each of these points has not been included in the surface modelling, but its deviation from the model has been calculated and presented with the absolute value of the true error of less than 5 cm.

All that leads to the conclusion that very optimistic results have been obtained with regards to the facts known in advance. Hence, providing that the average accuracy of GPS points in the homogeneous field is smaller or equal to 3 cm and that the average accuracy of node benchmarks amounts up to 1.41 cm, and that it can have maximum amounts of even 5.27 cm in lower accuracy orders, the obtained average internal model accuracy of 5 cm is a good indicator of the accuracy of selected models. One also has to take into consideration the fact that 16 (59.3%) points have been connected to benchmarks, and the rest of 11 (40.8%) points have been connected to only one benchmark, including also two fundamental points taken over (1037-Horvati and 1018-Sesvetski Kraljevec). The presented models transformed by means of Taylor's polynomial are very good approximation functions that can be used to define the model surface trend and also to complete the surface model, i.e. it is recommendable to use it for the first adjustments.

Table 6. External accuracy estimation for model 2.

	*FN*310		*FN*312		*FN*318	
	Number of GPS Points	%	Number of GPS Points	%	Number of GPS Points	%
$< 1\sigma$	12	44.4	16	59.3	16	61.5
$(1\sigma, 2\sigma)$	10	37.0	8	29.6	6	23.1
$(2\sigma, 3\sigma)$	5	18.5	3	11.1	4	15.4
$> 3\sigma$	0	0.0	0	0.0	0	0.0
Sum	27	100.0	27	100.0	26	100.0

2.2. *dN* Modelling with Non-Parametric Algorithms

In the previous chapter, the modelling of the surface with polynomials has been presented. The simplicity and general acceptability of the method, as well as the accuracy estimation measures developed for the models are some of the advantages as opposed to the surface modelling with non-parametric algorithms (*Analytical graphics package TableCurve3D, http://www.systat.com/products/TableCurve3D*). Still, any algorithm developed for the purpose of modelling the surface is based on the interpolation with a parametric function, mostly the polynomial of the lower order. The specific characteristic of these non-parametric algorithms is that they use various functions on various parts of the surface, adjusted to the set of data of a surface part in the best possible way. Such surface function is not continuous and it is not differentiable, but it has got interruptions in the points presenting the initial data. However, the modelling of the surface by means of applying such special algorithms describes the real model much better, especially the small local irregularities (*Sambridge et al., 1995*).

Such especially developed models can be used for the interpolation of the topographic data connected with the gravity field acceleration, magnetic field, etc. The application of a single model depends on the data, i.e. on the reference or nodal points. Hence, on the basis of the known coordinate values (X, Y, Z) of discrete points regularly or irregularly distributed in an area, the values of the Z coordinate of all other points can be calculated by means of interpolation (*Akima, 1996*). The distribution of nodal points in an area can be in the form of a network, regularly or irregularly distributed data. The nodal points can be more dense in some area, but in some other area they can be distributed with lower density, which depends on the collected data that one has at disposal. Depending on the nodal point schedule, adequate models will be applied, describing the given area geometrically with certain advantages and disadvantages (*Akima, 1978*).

In order to model the data in an area as good as possible, the so called "local" methods of interpolation are used that describe the physical reality much better on the basis of empirical data (*Sambridge et al., 1995*). Such local methods are most often based on the separation of the observed area of discrete values into smaller parts, the so called cells, i.e. on the procedure of developing a unique group of geometric figures, e.g. triangles, rectangles, irregular polygons.

The nodal points can be contained within the geometric figures or can be part of them, as is the case with vertices or line segments. Over such a smaller area, the method of interpolation is applied then with a function whose parameters are usually determined on the basis of the nodal point's values of the geometric figure area (*Sambridge et al., 1995*). The problem of the local interpolation is getting more complicated when the distribution of nodal points is of irregular pattern and density (scattered data). One should pay the greatest attention to two problems: in which way to develop the system of geometric figures, i.e. which of the developed groups of geometric figures is the most convenient, and how to find the fastest way to a geometric figure containing the requested point of interpolation. The part of the mathematics dealing with developing and analyzing algorithms for the purpose of solving these problems (space geometry) is called Computational Geometry. It is also present in the field of natural sciences, physics, as well as technical and information sciences and it is to be found in all problems connected with geometry (*Computational Geometry Pages, http://compgeom.cs.uiuc.edu/*

~jeffe/compgeom). It is also applied in a very wide area of geosciences, and applied in surface parameterization used for presenting the Earth's surface with a developed group of geometric figures. The basic geometric figures that are most often used are known by the name Vernoi diagrams and Delaunay triangulation (*Renka, 1996*).

The simplest case of interpolation is the linear function. However, the greatest disadvantage of the linear interpolation is the discontinuity phenomenon, seen already in the first derivation of the function in the points along the triangle's sides, and in vertices, i.e. nodal points. All other derivations within every triangle are also equal to zero. If the function with which some values are interpolated is derivable and if, at the same time, there are also second and third derivations, we speak about continuous surface defined with these functions (*Erol and Celik, unpublished results*).

Method of Natural Neighbours Interpolation

The described method of local interpolation is based on the idea of the "influence" of the closest nodal points encircling the point that needs to be interpolated in a certain local area to the value of interpolation (*Sambridge et al., 1995*). It is known as Natural Neighbours Interpolation method. The concept of this method is very simple and is based on looking for those data that will have the most similar characteristics and known behaviour. The datum having the most similar characteristics of the points that is to be interpolated is the closest "neighbor" and it is presumed that it will behave similarly. The questions as how to define which nodal points are the closest neighbours and how to find them quickly have been solved with various algorithms (*Sambridge et al., 1995*).

The natural neighbours of any nodal point are contained in Veroni's cells or equivalently, those nodal points that are connected with the sides of Delaney's triangles *(Triangulation, http://www.math.utah.edu/~alfeld/MDS/triangulation.html*). The most important characteristic of the definition of natural neighbours is that they present a group of the closest nodal points around some points. The number and the position, i.e. local distribution of nodal points will affect the definition of some point. Thus, the points in some areas will be surrounded by more, and in some by less natural neighbours. In the same way, the distance between natural neighbours and a point will be larger or smaller (*Sambridge et al., 1995*). One can imagine that the natural neighbours of any point in a defined area are contained in a unique group of nodal points defining the neighbourhood of a point in a plane. If the distance among nodal points is great in some parts, or the distribution of points is highly anisotropic (isotropic - a characteristic of some data to indicate equal physical characteristics in various directions), then a set of natural neighbours will describe the characteristics of this area, and still present the best group of the closest surrounding nodal points. Such approach presents the best basis for the application of the local interpolation that can be expressed as (*Sambridge et al., 1995*)

$$f\left(y,x\right)=\sum_{i=1}^{n}\varphi_i\left(y,x\right)f_i\ , \tag{15}$$

where $f(y, x)$ is the value of the function in the point of interpolation, f_i ($i = 1, \ldots, n$) are the values of data in n natural neighbouring nodal points (y, x), and φ_i ($i = 1, \ldots, n$) are the weights associated to some nodal point defined as a natural neighbour. The harmony of interpolation depends as well on the function of defining the weight φ_i. However, since the sum in the previous formula is only the sum of natural neighbouring nodal points, then the interpolation will be local regardless of how φ_i is defined. Apart from that, the size and form of the defined area will be adjusted to the nodal point's density. Some methods of determining the weights are based on the amount of the surface of Vernoi's cells. Such methods of looking for natural neighbours and of local interpolation in this area result in a continuous first derivation of the interpolation function in all points except for nodal points, and it is recommendable to use it with very irregular distribution of points (*Sambridge et al., 1995*). The most important characteristics of these algorithms are the following:

- The function value in the nodal point is equal to the measured value, i.e. to the input data in nodal points;
- Interpolation is local (the point is influenced only by its neighbouring nodal points);
- Derivations of interpolation functions are continuous in the entire area, apart from the nodal points.

Models of Transformation with Non-Parametric Algorithms

The models of transforming ellipsoidal height of GPS points into the normal orthometic height by means of the interpolation with Watson and Loess algorithms have been tested. Generally speaking, the modelling of surfaces with non-parametric algorithms is recommended in the case when it is necessary to make good approximation of the local irregularities and when the trend of the input data behaviour is not clearly expressed.

The author of the Watson algorithm is David Watson. The algorithm has one of the most sophisticated methods of selecting the neighbouring points, i.e. nodal points, but also the worst interpolation procedure. The Watson algorithm uses the method of the closest natural neighbour that is based on looking for the neighbouring nodal points by means of circles. In order to interpolate heights, the Watson algorithm uses the weight arithmetic mean of the values of the z argument (the height in case of dN) of the neighbouring nodal points. The procedure of interpolation and extrapolation is linear which is not good if it is necessary to model area trends among the nodal points with the higher order of interpolation function. The method of extrapolation is primitive and of low accuracy.

The Loess algorithm is the only algorithm that uses the method of data smoothing before the method of interpolating irregularly distributed nodal points, and all other methods have only the possibility of exact interpolation. The author of this algorithm that has never been published is Ron Brown, but the basic information about the Loess algorithm can be found in *Cleveland (1993).* The 3D Loess algorithm actually consists of two algorithms that include the procedure of data smoothing and the procedure of interpolation: It has been developed especially for the TableCurve3D program package. Hence, the Loess algorithm applies first the technique of interpolation with the method of Dealunay's triangulation on smoothed data. For that reason, the same reduced values of undulation (dN) that were there before the modelling, have not been saved for empirical input data, i.e. nodal points. The Loess procedures consist of fitting three parametric linear, six parametric square or ten parametric cubic functions using the method of the closest natural neighbour. Hence, three models of fitting are used. The first order model is the plane ($y = a + bx + cy$), the second order model is the second degree Taylor's polynomial ($y = a + bx + cy + dx^2 + ey^2 + fxy$)and the third order is the third degree Taylor's polynomial ($y = \mathrm{a} + bx + cy + dx^2 + ey^2 + fxy + gx^3 + hy^3 + ix^2y + jxy^2$). Referring to the degree and the number of polynomial parameters, the number of the neighbouring nodal points used in local fitting is determined. The smoothed data are used as nodal points in the interpolation of Renka I (*Renka, 1996*) that is based on the method of Dealunay's triangulation interpolation. This algorithm enables continuous first and second partial derivations in the procedure of interpolation and of extrapolation. The good side of this algorithm is that it has got one of the best developed data smoothing methods. The speed of algorithm is improved because the interpolation is developed on smoothed data, although the method of Dealyunay's triangulation is used.

Accuracy Estimation of Individual Non-Parametric Transformation

The external accuracy of the transformation model with non-parametric algorithms has been tested in the same way as in the previous chapter.

Table 7 shows the percentage values of the number of GPS points given, along with the size of their true value belonging to a certain interval. It can be noticed that almost 50% of the GPS control points have absolute residual values smaller than 4 cm. The number of points having the deviation from the Loess model smaller than 6 cm reaches as much as 70%.

The next two tables (Table 8 and 9) present the estimation of internal and external accuracy of individual non-parametric model.

The minimum and maximum absolute residual values, residual square sum, and of variances and standard deviation for the models Watson and Loess are calculated in Tables 8 and 9. One can see immediately that the given values for the Watson model are equal to 0.00 cm. The reason lies in the definition of the algorithm that fixes all input data, i.e. it does not change the *dN* value, unlike the Loess algorithm where the input data are first smoothed. The minimum absolute value of residuals with Loess model is 0.06 cm, and the maximum value reaches the amount of 8.66 cm for the GPS point 1840-Štefanovec. GPS point 1840-Štefanovec is located at a higher altitude in the area of

Table 7. The percentage of the number of points having the accuracy in the interval of 0 to 6 cm.

σ	Watson		Loess	
	Number of Points	%	Number of Points	%
< 1 cm	5	18.5	4	14.8
< 2 cm	6	22.2	9	33.3
< 3 cm	9	33.3	14	51.9
< 4 cm	13	48.1	15	55.6
< 5 cm	15	55.6	18	66.7
< 6 cm	18	66.7	19	70.4

Table 8. Estimation of internal accuracy.

	Watson [cm]	Loess [cm]
MinAbs(r) (Number of GPS Point)	0.00	0.06 (1237)
MaxAbs(r) (Number of GPS Point)	0.00	8.66 (1840)
Σr^2	0.00	2.55
σ^2	0.00	0.09
σ	0.00	3.07

Table 9. Estimation of external accuracy.

	Watson [cm]	Loess [cm]
MinAbs(r) (Number of GPS Point)	0.09 (4026)	0.04 (2795)
MaxAbs(r) (Number of GPS Point)	26.60 (1331)	18.70 (1018)
$\Sigma\sigma^2$	0.00	2.42
$\Sigma\sigma^2/27$	0.00	3.05

Medvednica and connected to only one benchmark 887 of the levelling figure 630, II. Polygon of II. NVT built into the sewers of the vertically stabilized bridge.

Provided that the combinations of 27 various models slightly differ from each other, one can compare the deviations of 27 control points (Table 9). The diversity of algorithms can be noticed in the analysis of the deviation as well where different points appear with both algorithms acquiring maximum and minimum values. The point 1331-Sljeme has the largest absolute residual value in the Watson model, and in the Loess model it is the case with the point 1018-Sesvetski Kraljevec. Both points are extrapolated, i.e. they appear in edge areas, with the GPS point 1331-Sljeme being the one with the highest altitude.

Table 10 presents external accuracy estimates of the Watson and Loess models. The comparisons are presented as related to the σ_1 and σ_2 standard deviations. Standard deviation σ_1 has been computed on the basis of the deviations of the measured values from the dN_{model} values using the model determined at all 27 points. Since the Watson model fixes input data, the deviations are equal to zero, and therefore, this accuracy estimate is very convenient, because the deviations of all points are smaller than 1σ. However, in order to define the accuracy of this model more realistically, the analysis has been made in relation to the standard deviation σ_2, that is calculated from 27 residual values of GPS control points calculated with the Watson and Loess model (*Gucek, 2005*).

The residuals of GPS points have been calculated in the already described way, through the difference of the measured value and the one obtained from the model. Then

Table 10. Comparison of external accuracy of the models Watson and Loess.

σ	Watson $\sigma_1 = 0.00$ cm		Watson $\sigma_2 = 8.23$ cm		Loess $\sigma_1 = 3.07$ cm		Loess $\sigma_1 = 6.90$ cm	
	Number of GPS Points	%	Number of GPS Points	%	Number of GPS Points	%	Number of GPS Points	%
$< 1\sigma$	27	100.0	21	77.8	14	51.9	19	70.4
$(1\sigma, 2\sigma)$	0	0.0	5	18.5	5	18.5	6	22.2
$(2\sigma, 3\sigma)$	0	0.0	1	3.7	7	25.9	2	7.4
$> 3\sigma$	0	0.0	0	0.0	1	3.7	0	0.0
Sum	27	100.0	27	100.0	27	100.0	27	100.0

the σ_2 standard deviation has been computed on the basis of the residual values for the GPS control points obtained from 27 models that differ by about 7% of input data.

Thus obtained results are presented in Table 10 in columns with the marks Watson-σ_2 and Loess-σ_2. It is obvious that 78%, i.e. 21 GPS points deviate from the Watson model for less than 8.23 cm ($1\sigma_2$), and that more than 70% of GPS control points deviate from the Loess model for less than 6.90 cm ($1\sigma_2$). A very important fact is also that the residual of only one GPS control point (1331-Sljeme) amounting in the interval [$2\sigma_2$, $3\sigma_2$] for the model Watson, and with the Loess model it happens with two GPS points (1018-Sesvetski Kraljevec and 1840-Štefanovec). The above mentioned points have larger residual amounts recorded.

There are a few conclusions to be made on the basis of the whole analysis connected with the modelling of the surface using Watson and Loess algorithms, taking into account also previous theoretical explanations. The data from Table 10 speak in favour of the Watson algorithm. Namely, in the Watson algorithm almost 78% of GPS control points have the residual amounts smaller than $1\sigma_2$ (8.23 cm), and in the Loess algorithm this information is equal to 70.4%. The difference of 8% in favour of the Watson algorithm is only 2 points. However, one should take into consideration the comparison of the standard deviation amount that is bigger in the Watson algorithm where $1\sigma_2 = 8.23$ cm while in the Loess algorithm, this amount is smaller and reads $1\sigma_2 = 6.90$ cm. The best comparison of residuals is presented in Table 7 where it is to be seen that although 18% of GPS control points in the Watson algorithm have absolute deviation values smaller than 1 cm, finally 55.6% or 18 GPS control points have got the amount of absolute residual values smaller than 5 cm in the Loess algorithm, as related to the 15 GPS control points or 48.1% in the Watson algorithm. One should also take into consideration the size of the interval between the minimum and maximum values that runs up to 25.51 cm in modelling with the Watson algorithm, and 18.66 cm in the Loess algorithm.

3. CONCLUSIONS AND RECOMMENDATIONS

The empirical data used for modelling were reduced undulation values dN of GPS/levelling points, which are considered as quantities without errors. Within the frame of dN modelling with parametric functions the approximation surfaces were obtained on the basis of three polynomials: $FN310$, $FN312$ and $FN318$. The modelling with non-parametric Watson and Loess algorithms was also done. The $FN318$ and Loess models yielded the best results of reduced undulation values of dN, and therefore Table 11 gives an overview of the normal orthometric heights of GPS/levelling points determined in measurements (H_{HVRS71}) and defined by the transformation using the $FN318$ and Loess models (H_{FN318}, H_{Loess}), as well as the differences ($\Delta H_{FN318} = H_{FN318} - H_{HVRS71}$, $\Delta H_{Loess} = H_{Loess} - H_{HVRS71}$).

Table 11. Overview and comparison of GPS points heights.

GPS	H_{HVRS71} [m]	H_{FN318} [m]	ΔH_{FN318} [cm]	H_{Loess} [m]	ΔH_{Loess} [cm]
1018	195.1145	195.1519	3.74	195.1282	1.37
1037	240.6171	240.6197	0.26	240.6324	1.53
1237	282.9077	282.9279	2.02	282.9083	0.06
1331	1004.9174	1004.9163	−0.11	1004.9210	0.34
1476	223.3784	223.3405	−3.79	223.3656	−1.29
1781	266.8707	266.9468	7.61	266.9382	6.75
1840	271.3994	271.3157	−8.37	271.3127	−8.66
1982	221.1006	221.1649	6.43	221.1491	4.86
2573	216.7555	216.7201	−3.54	216.7275	−2.80
2795	229.6351	229.6302	−0.49	229.6372	0.21
3010	216.4851	216.4895	0.44	216.4843	−0.08
3040	213.1432	213.1361	−0.71	213.1380	−0.52
3080	217.8644	217.9057	4.13	217.8785	1.41
3215	203.2095	203.1908	−1.87	203.2184	0.89
3453	197.1920	197.1388	−5.31	197.1823	−0.97
3961	214.7853	214.7746	−1.06	214.7754	−0.98
4026	201.7758	201.7527	−2.31	201.7895	1.37
4204	204.6601	204.6692	0.91	204.6523	−0.78
4497	202.0982	202.0957	−0.25	202.0722	−2.61
4501	203.8965	204.0016	10.51	203.9715	7.50
4578	212.5879	212.5574	−3.04	212.5582	−2.97
4750	288.4420	288.4186	−2.34	288.4270	−1.51
4862	214.0122	213.9776	−3.46	213.9774	−3.48
4870	236.6880	236.7170	2.90	236.6961	0.82
4978	272.8196	272.8068	−1.28	272.8228	0.32
4992	233.5037	233.5114	0.78	233.5156	1.20
5081	382.3513	382.3335	−1.78	382.3621	1.08

The analysis and estimation of accuracy for each single model were determined on the basis of deviations of measured values of GPS/levelling points from the computed values obtained from the models. Very good and high-quality results were obtained referring to the analysis and the presented facts. First of all, assuming that the average accuracy of the height component of GPS points in homogeneous field is 3 cm and that the average accuracy of nodal benchmarks is between 1.41 cm and 5.27 cm, the obtained internal accuracy of individual models is obtained and recorded in the standard deviation of 3.07 cm to 5.11 cm. The highest internal accuracy of parametric transformation models is achieved with the *FN*318 polynomial, having the standard deviation of 4.99 cm, and the surface modelling with the Loess algorithm yielded the best results of the accuracy estimation running up to 3.07 cm. At the same time, the external accuracy of the models was given by cross validation method. In almost all models, about 70% of GPS/levelling control points have the absolute deviation value from the models smaller than 6 cm. The largest percentage of 74.1% (20 points) in the case of parametric functions is found in the *FN*312 model, and not the non-parametric Loess algorithm having 70.4% (19 points).

Polynomials are very good functions of approximation that can be used for defining the trends of various surface models, i.e. it is recommendable to use them for the first approximations. However, taking into consideration all parameters, the modelling of the surface that would be used for transforming the ellipsoidal height into normal orthometric height speaks in favour of the Loess algorithm. Theoretically, the positive sides of this algorithm can already be seen because it is obviously that this algorithm is good for modelling the type of data that are irregularly distributed over the whole area of modelling. Still, its disadvantage should also be pointed out. Namely, the input data do not have fixed original values, but have changed in the interval of as much as 8 cm. The fact that the normal orthometric height of GPS points determined with measurements will not be determined again from the model indicates that this disadvantage does not present a problem in practice.

On the basis of the tested models applied only within the frame of the test area, a few essential facts should be taken into consideration in future studies and projects. The accuracy of any selected model will largely depend on the accuracy of the input data, the terrain configuration and the distribution of GPS/levelling points. The selection of points should not always concentrate on the raster and equally cover the area, but should also, as much as possible, encompass the characteristic topology, i.e. all recesses and elevations. In hilly and irregular areas, the density of the points must be larger than in the planes. The density of GPS/levelling points is also important, and the accuracy of the models certainly depends on it. It is therefore recommended to test the influence of a number of GPS/levelling points in future analyses on the accuracy of the selected transformation model, which will affect the cost-effectiveness of the transformation. Hence, the following question needs to be answered: what is the highest price with regard to the required transformation accuracy? It is evident that in the future projects, GPS points should be used for such testing having higher-quality stabilization as related to the GPS points in homogeneous field, and they should be, if possible, connected to the minimum of two benchmarks in the highest order levelling. In this case, all points of the basic GPS network of the city of Zagreb should be included in the testing and the accuracy of input data should be calculated taking into account the individual accuracy of ellipsoidal heights of GPS points, normal orthometric benchmark heights and measurement accuracy. In this way the accuracy of individual GPS/levelling points in the model could be indicated through the weight associated to each input information. The determination of the normal orthometric height from the model by using the method of extrapolation should be avoided. Therefore, GPS/levelling points from the area outside the one being the object of modelling should be included for specific areas.

Before selecting a transformation model itself, all input data should be tested in order to eliminate gross errors, but at the same time it is recommendable to test the model itself statistically, especially the parameters when dealing with exact mathematical parametric functions. The testing conducted for the purpose in this paper has shown that the model of the surface approximation in which the interpolation is applied on smoothed data, has yielded better accuracy results, which can be seen in the application of the Loess algorithm. It should be underlined that it is most convenient to check the external accuracy of the transformation model on an independent group of data.

Finally, it is extremely important to point out that the selection of the model must depend on a few elements: first of all on the required accuracy, practicality of application, practical use and unavoidably, cost-effectiveness.

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