Abstract for 24th IWWWFB, Saint Petersbourg, Russia, 2009

Some aspects of hydrostatic restoring for elastic bodies

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Introduction

The paper deals with the evaluation of the linear hydrostatic restoring matrix for elastic body. In spite of quite important work on the subject (eg see [1, 2, 3, 4, 5, 6, 7]), the problem still seems to not be fully clear and different expressions proposed in the literature do not match each other!? On the other hand the application of, what seems to be the correct method, leads to some strange results for the internal loads!? The main purpose of the paper is to discuss and compare different methods.

Direct perturbation method

Before continuing, let us just recall the definition of the restoring coefficient which can be briefly stated as the ratio in between the reaction force and the displacement which produces it when the body is moved from initially equilibrated position in calm water. This means that the hydrostatic restoring will be composed not only of the pure hydrostatic pressure part but from all the forces which participate to the initial equilibrium of the body (gravity, concentrated external forces, ...) General situation is shown in Figure 1 (bold letters are used to denote the vector quantities). The instantaneous position of one

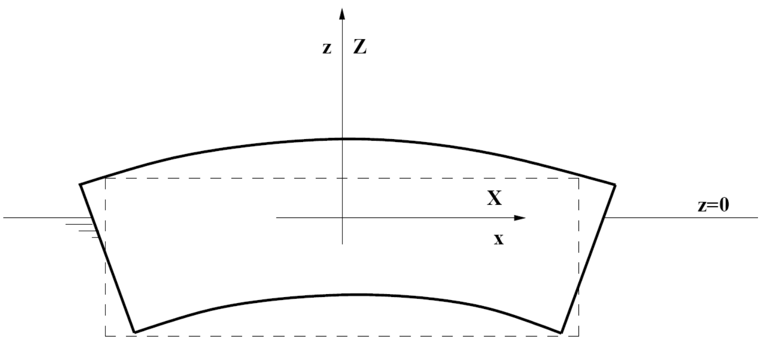

Figure 1: *Generalized body motion.*

point on the body is described by the vector $\boldsymbol{r}$ and the corresponding position at rest by the vector $\boldsymbol{R}$:

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \qquad , \qquad \boldsymbol{R} = X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k} \tag{1}$$

The displacement vector for mode j is described by the vector $\boldsymbol{h}^j$:

$$\boldsymbol{h}^j = h_X^j\boldsymbol{i} + h_Y^j\boldsymbol{j} + h_Z^j\boldsymbol{k} \tag{2}$$

where h_X^j, h_Y^j, h_Z^j are the arbitrary functions of X, Y, Z.
Within these notations, the following relation is valid:

$$\boldsymbol{r} = \boldsymbol{R} + \boldsymbol{h}^j \tag{3}$$

We define the generalized hydrostatic pressure force, on mode i (note that the coefficient ϱg is omitted throughout whole the paper):

$$\tilde{\boldsymbol{F}}_{ij}^H = -\iint_{\tilde{S}_B} z\tilde{\boldsymbol{h}}^i\tilde{\boldsymbol{n}}d\tilde{S} \tag{4}$$

where the tilde sign "$\tilde{}$" denotes the instantaneous value of the corresponding quantity.
The final goal of the analysis is to extract, from the above equation, the leading order term with respect to the small modal displacement $\boldsymbol{h}^j$. In order to do that, we need to rewrite the above expression at the initial body position. We write:

$$\tilde{\boldsymbol{F}}_{ij}^H = -\iint_{S_B+\delta S_B} (Z+\delta Z)(\boldsymbol{h}^i + \delta\boldsymbol{h}^i)[\boldsymbol{n}dS + \delta(\boldsymbol{n}dS)] \tag{5}$$

where δ denotes the change of the corresponding quantity due to the body motion/distortion.
It can be shown that the integral over δS_B is of higher order so that only the integral over the initial wetted position S_B remains. The change of each quantity can be obtained using the notion of the deformation gradient [the subscript $_X$ means that the derivatives are to be performed with respect to the coordinate system (X,Y,Z)]:

$$\underline{\underline{\nabla_X \boldsymbol{h}^j}} = \begin{bmatrix} \dfrac{\partial h_X^j}{\partial X} & \dfrac{\partial h_X^j}{\partial Y} & \dfrac{\partial h_X^j}{\partial Z} \\ \dfrac{\partial h_Y^j}{\partial X} & \dfrac{\partial h_Y^j}{\partial Y} & \dfrac{\partial h_Y^j}{\partial Z} \\ \dfrac{\partial h_Z^j}{\partial X} & \dfrac{\partial h_Z^j}{\partial Y} & \dfrac{\partial h_Z^j}{\partial Z} \end{bmatrix} \tag{6}$$

The differential change of the different quantities become:

$$\delta Z = \underline{\underline{\nabla_X(Z\boldsymbol{k})}}\cdot \boldsymbol{h}^j = h_Z^j \quad , \quad \delta\boldsymbol{h}_i = \underline{\underline{\nabla_X \boldsymbol{h}^i}}\cdot\boldsymbol{h}^j \quad , \quad \delta(\boldsymbol{n}dS) = \nabla_X\boldsymbol{h}^j\ \boldsymbol{n} - (\underline{\underline{\nabla_X\boldsymbol{h}^j}})^T\cdot\boldsymbol{n} \tag{7}$$

where overscript ^{T} denotes the transpoze operation.
The final expression for the generalized restoring coefficient becomes:

$$C_{ij}^H = \iint_{S_B} \left\{ h_Z^j\boldsymbol{h}^i\cdot\boldsymbol{n} + Z(\underline{\underline{\nabla_X\boldsymbol{h}^i}}\cdot\boldsymbol{h}^j)\cdot\boldsymbol{n} + Z(\nabla_X\boldsymbol{h}^j)\boldsymbol{h}^i\cdot\boldsymbol{n} - Z[(\underline{\underline{\nabla_X\boldsymbol{h}^j}})^T\cdot\boldsymbol{n}]\cdot\boldsymbol{h}^i \right\} dS \tag{8}$$

By using the identity $(\underline{\underline{\nabla_X\boldsymbol{h}^j}}^T\cdot\boldsymbol{n})\cdot\boldsymbol{h}^i = (\underline{\underline{\nabla_X\boldsymbol{h}^j}}\cdot\boldsymbol{h}^i)\cdot\boldsymbol{n}$ the above expression can be rewritten as:

$$C_{ij}^H = \iint_{S_B} \left\{ Z[\nabla_X\boldsymbol{h}^j\boldsymbol{h}^i\cdot\boldsymbol{n} + (\underline{\underline{\nabla_X\boldsymbol{h}^i}}\cdot\boldsymbol{h}^j - \underline{\underline{\nabla_X\boldsymbol{h}^j}}\cdot\boldsymbol{h}^i)\cdot\boldsymbol{n}] + h_Z^j\boldsymbol{h}^i\boldsymbol{n} \right\} dS \tag{9}$$

Molin's formulation

Molin's [3] used quite different method involving the integral transformations in order to represent the restoring in terms of the volume integrals. First we rewrite the general hydrostatic effort in the form:

$$\tilde{\boldsymbol{F}}_{ij}^H = \iint_{\tilde{S}_B} z\tilde{\boldsymbol{h}}^i\tilde{\boldsymbol{n}}d\tilde{S} = \iint_{\tilde{S}_B+\tilde{S}_F} z\tilde{\boldsymbol{h}}^i\tilde{\boldsymbol{n}}d\tilde{S} - \iint_{\tilde{S}_F} z\tilde{\boldsymbol{h}}^i\tilde{\boldsymbol{n}}d\tilde{S} = \tilde{\boldsymbol{F}}_{ij}^{H1} + \tilde{\boldsymbol{F}}_{ij}^{H2} \tag{10}$$

where $\tilde{S}_F$ denotes the instantaneous waterline surface.
The first part of the generalized force $\tilde{\boldsymbol{F}}_{ij}^{H1}$, is transformed into the volume integral:

$$\tilde{\boldsymbol{F}}_{ij}^{H1} = \iiint_{\tilde{V}_B} \nabla_x(z\boldsymbol{h}^i)d\tilde{V} = \iiint_{\tilde{V}_B} (z\nabla_x\boldsymbol{h}^i + h_z^i)d\tilde{V} \tag{11}$$

In order to transform the above integral from the instantaneous position $\tilde{V}(x,y,z)$ into the initial one $V(X,Y,Z)$, the following relations are used:

$$d\tilde{V} = (1+\nabla_X\boldsymbol{h}^j)dV \quad , \quad z = Z + h_Z^j \quad , \quad \tilde{\boldsymbol{h}}^i = \boldsymbol{h}^i + \underline{\underline{\nabla_X\boldsymbol{h}^i}}\cdot\boldsymbol{h}^j \tag{12}$$

$$\nabla_x\tilde{\boldsymbol{h}}^i = \nabla_X\boldsymbol{h}^i + \boldsymbol{h}^j\cdot\nabla_X(\nabla_X\boldsymbol{h}^i) \quad , \quad h_z^i = h_Z^i + \boldsymbol{h}^j\cdot\nabla_X h_Z^i \tag{13}$$

After inserting the above expressions into (11), the following expression is obtained at leading order:

$$C_{ij}^{H1} = \iiint_{V_B} \left\{ Z[\nabla_X\boldsymbol{h}^i\nabla_X\boldsymbol{h}^j + \boldsymbol{h}^j\cdot\nabla_X(\nabla_X\boldsymbol{h}^i)] + h_Z^j\nabla_X\boldsymbol{h}^i + h_Z^i\nabla_X\boldsymbol{h}^j + \boldsymbol{h}^j\cdot\nabla_X h_Z^i \right\} dV \tag{14}$$

At the same time, the leading order term of the second part of the generalized hydrostatic force is easily obtained as:

$$C_{ij}^{H2} = -\iint_{S_F} h_Z^i h_Z^j dS \tag{15}$$

Newman's formulation

In Newman's formulation [4], the restoring coefficient is defined by the following expression:

$$C_{ij}^{H} = \iint_{\tilde{S}_B} z\tilde{\boldsymbol{h}}^i \tilde{\boldsymbol{n}} dS - \iint_{S_B} z\boldsymbol{h}^i \boldsymbol{n} dS = \iiint_{\Omega} \nabla_X (Z\boldsymbol{h}^i) d\Omega = \iiint_{\Omega} (Z\nabla_X \boldsymbol{h}^i + h_Z^i) d\Omega \tag{16}$$

where Ω denotes the volume in between the instantaneous wetted surface $\tilde{S}_B$ and the initial one S_B. Under the small displacement assumptions we can write $d\Omega = \boldsymbol{h}^j \boldsymbol{n} dS$ so that the final expression for the restoring coefficient becomes:

$$C_{ij}^{H} = \iint_{S_B} (Z\nabla_X \boldsymbol{h}^i + h_Z^i)\boldsymbol{h}^j \boldsymbol{n} dS \tag{17}$$

Equivalence of different expressions

Newman to Molin

In order to compare Newman's formulation to Molin's formulation, first we subdivide the expression (17) in the following way:

$$C_{ij}^{H} = \iint_{S_B} Z\nabla_X \boldsymbol{h}^i \boldsymbol{h}^j \boldsymbol{n} dS + \iint_{S_B} h_Z^i \boldsymbol{h}^j \boldsymbol{n} dS = C_{ij}^{Ha} + C_{ij}^{Hb} \tag{18}$$

The first part is now transformed into volume integral:

$$\begin{aligned} C_{ij}^{Ha} &= \iint_{S_B+S_F} Z\nabla_X \boldsymbol{h}^i \boldsymbol{h}^j \boldsymbol{n} dS = \iiint_V \nabla_X (Z\nabla_X \boldsymbol{h}^i \boldsymbol{h}^j) dV \\ &= \iiint_V \left\{ Z[\nabla_X \boldsymbol{h}^i \nabla_X \boldsymbol{h}^j + \boldsymbol{h}^j \nabla_X (\nabla_X \boldsymbol{h}^i)] + h_Z^j \nabla_X \boldsymbol{h}^i \right\} dV \end{aligned} \tag{19}$$

The second integral is transformed into:

$$\begin{aligned} C_{ij}^{Hb} &= \iint_{S_B+S_F} h_Z^i \boldsymbol{h}^j \boldsymbol{n} dS - \iint_{S_F} h_Z^i h_Z^j dS = \iiint_V \nabla_X (h_Z^i \boldsymbol{h}^j) dV - \iint_{S_F} h_Z^i h_Z^j dS \\ &= \iiint_V (h_Z^i \nabla_X \boldsymbol{h}^j + \nabla_X h_Z^i \boldsymbol{h}^j) dV - \iint_{S_F} h_Z^i h_Z^j dS \end{aligned} \tag{20}$$

It is now easy to see that $C_{ij}^{H} = C_{ij}^{H1} + C_{ij}^{H2} = C_{ij}^{Ha} + C_{ij}^{Hb}$.

Direct to Molin

The original expression (9) is subdivided into two parts:

$$C_{ij}^{H} = \iint_{S_B} Z[\nabla_X \boldsymbol{h}^j \boldsymbol{h}^i \cdot \boldsymbol{n} + (\underline{\underline{\nabla_X \boldsymbol{h}^i}} \cdot \boldsymbol{h}^j - \underline{\underline{\nabla_X \boldsymbol{h}^j}} \cdot \boldsymbol{h}^i) \cdot \boldsymbol{n}] dS + \iint_{S_B} h_Z^j \boldsymbol{h}^i \boldsymbol{n} dS = C_{ij}^{Hc} + C_{ij}^{Hd} \tag{21}$$

As in the previous section, the first integral is transformed into the volume integral:

$$C_{ij}^{Hc} = \iiint_V \nabla_X \left\{ Z[\nabla_X \boldsymbol{h}^j \boldsymbol{h}^i \cdot \boldsymbol{n} + (\underline{\underline{\nabla_X \boldsymbol{h}^i}} \cdot \boldsymbol{h}^j - \underline{\underline{\nabla_X \boldsymbol{h}^j}} \cdot \boldsymbol{h}^i) \cdot \boldsymbol{n}] \right\} dV \tag{22}$$

After rearranging different terms, the above expression can be rewritten in the following form:

$$C_{ij}^{Hc} = \iint_{S_B+S_F} \left\{ Z[\nabla_X \boldsymbol{h}^i \nabla_X \boldsymbol{h}^j + \boldsymbol{h}^j \nabla_X (\nabla_X \boldsymbol{h}^i)] + h_Z^i \nabla_X \boldsymbol{h}^j + \boldsymbol{h}^j \nabla_X h_Z^i - \boldsymbol{h}_i \nabla_X h_Z^j \right\} dV \tag{23}$$

At the same time, the second term is rearranged into:

$$\begin{aligned} C_{ij}^{Hd} &= \iint_{S_B+S_F} h_Z^j \boldsymbol{h}^i \boldsymbol{n} dS - \iint_{S_F} h_Z^j h_Z^i dS = \iiint_V \nabla_X (h_Z^j \boldsymbol{h}^i) dV - \iint_{S_F} h_Z^i h_Z^j dS \\ &= \iiint_V (h_Z^j \nabla_X \boldsymbol{h}^i + \nabla_X h_Z^j \boldsymbol{h}^i) dV - \iint_{S_F} h_Z^i h_Z^j dS \end{aligned} \tag{24}$$

By summing up the two terms we can easily show that $C_{ij}^H = C_{ij}^{H1} + C_{ij}^{H2} = C_{ij}^{Ha} + C_{ij}^{Hb} = C_{ij}^{Hc} + C_{ij}^{Hd}$.

Huang & Riggs formulation

Apparently, the Huang & Riggs [5] formulation is the same as the direct approach except that the term:

$$\delta \boldsymbol{h}_i = \underline{\underline{\nabla_X \boldsymbol{h}^i}} \cdot \boldsymbol{h}^j \tag{25}$$

is omitted. This makes the Huang & Riggs formulation different from the others.

Discussions

The above expressions represents the hydrostatic pressure part only and the gravity related part should be added in order to obtain the final expression for the restoring. This gravity related part should be the same for all the approaches, and can be derived in the following form:

$$C_{ij}^g = g \iiint_{V_B} (\boldsymbol{h}^j \nabla_X) h_Z^i dm \tag{26}$$

One possibility to check the validity of different formulations is to calculate the well known restoring matrix for rigid body modes of motions. Indeed, the six rigid body modes of motion can be defined as:

$$\boldsymbol{h}^1 = \boldsymbol{i} \quad , \quad \boldsymbol{h}^2 = \boldsymbol{j} \quad , \quad \boldsymbol{h}^3 = \boldsymbol{k} \quad , \quad \boldsymbol{h}^4 = \boldsymbol{i} \wedge (\boldsymbol{R} \wedge \boldsymbol{R}_G) \quad , \quad \boldsymbol{h}^5 = \boldsymbol{j} \wedge (\boldsymbol{R} \wedge \boldsymbol{R}_G) \quad , \quad \boldsymbol{h}^6 = \boldsymbol{k} \wedge (\boldsymbol{R} \wedge \boldsymbol{R}_G) \tag{27}$$

where $\boldsymbol{R}_G$ denotes the vector position of the center of gravity.
When applying the above discussed formulations, to these modal functions, the classical restoring matrix for rigid body is recovered by all the formulations except the one given by Huang & Riggs.
It is however not fully clear if the Huang & Riggs formulation should be compared directly to other formulations, since their formulation includes also some other terms such as the internal geometric stiffness. At the same time, the other formulations still have some problems in evaluating the internal loads!? All this will be discussed more in details at the Workshop.

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