

HEISENBERG DOUBLE VERSUS DEFORMED DERIVATIVES

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Two approaches to the tangent space of a noncommutative space whose coordinate algebra is the enveloping algebra of a Lie algebra are known: the Heisenberg double construction and the approach via deformed derivatives, usually defined by procedures involving orderings among noncommutative coordinates or equivalently involving realizations via formal differential operators. In an earlier work, we rephrased the deformed derivative approach introducing certain smash product algebra twisting a semicompleted Weyl algebra. We show here that the Heisenberg double in the Lie algebra case, is isomorphic to that product in a nontrivial way, involving a datum ϕ parametrizing the orderings or realizations in other approaches. This way, we show that the two different formalisms, used by different communities, for introducing the noncommutative phase space for the Lie algebra type noncommutative spaces are mathematically equivalent.

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1. Introduction

1.1. Noncommutative algebras and noncommutative geometry may play various roles in models of mathematical physics; for example describing quantum symmetry algebras. A special case of interest is when the noncommutative algebra is playing the role of the space-time of the theory, and is interpreted as a small deformation of the (commutative) 1-particle configuration space. If one wants to proceed toward developing field theory on such a space, it is beneficial to extend the deformation of the configuration space to a deformation of full phase space (symplectic manifold) of the theory. Deformed momentum space for the noncommutative configuration space whose coordinate algebra is the enveloping algebra of a finite-dimensional Lie algebra (also called Lie algebra type noncommutative spaces) has been studied recently in the mathematical physics literature^{1,3,5}, mainly in special cases, most notably variants of so-called κ -Minkowski space^{2,3,11,9,15}.

1.2. (Deformed derivative approach)

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1.2.1. (Notation) The algebras in the article are over a field \mathbf{k} of characteristic zero; both real and complex numbers appear in applications of the present formalism. We fix a finite dimensional Lie algebra \mathfrak{g} with basis $\hat{x}_1, \dots, \hat{x}_n$, which are also the generators of enveloping algebra $U(\mathfrak{g})$; the corresponding commuting generators of the symmetric algebra $S(\mathfrak{g})$ will be denoted x_1, \dots, x_n .

1.2.2. Some authors (e.g. 3,5,15) add to the (linear) enveloping algebra generators, the corresponding “deformed” partial derivatives $\partial^1, \dots, \partial^n$, which are the dual variables (in \mathfrak{g}^*), assumed to mutually commute. Then they seek for the consistent commutation relations of the form

$$[\hat{x}_i, \partial^j] = \phi_i^j, \quad i, j = 1, \dots, n, \quad (1)$$

where $\phi_j^i = \phi_j^i(\partial^1, \dots, \partial^n) \in \hat{S}(\mathfrak{g}^*)$ are formal power series and $\phi_j^i = \delta_j^i +$ higher order terms. By ”consistency” they mean that one quotients the free associative algebra product of $U(\mathfrak{g})$ and of the (commutative) formal power series ring in $\partial^1, \dots, \partial^n$ (the latter is isomorphic to the completion of polynomial ring in the dual variables $\hat{S}(\mathfrak{g}^*)$) by the commutation relations (1) and the restriction of the quotient map to each of the parts, $U(\mathfrak{g})$ and $\hat{S}(\mathfrak{g}^*)$ separately, has a zero kernel. The resulting quotient algebra generated by $\hat{x}_1, \dots, \hat{x}_m, \partial^1, \dots, \partial^n$ will be referred to as the **phase space algebra with the ϕ -deformed derivatives** (of the noncommutative algebra $U(\mathfrak{g})$). It follows from the Jacobi identities¹⁴ that this nondegeneracy condition for the matrix $(\phi_j^i)_{i,j=1,\dots,n}$ can be expressed by requiring that (ϕ_j^i) provides a solution to the system

$$\phi_j^l \frac{\partial}{\partial(\partial^l)}(\phi_i^k) - \phi_i^l \frac{\partial}{\partial(\partial^l)}(\phi_j^k) = C_{ij}^s \phi_s^k. \quad (2)$$

of formal differential equations (summation on repeated indices understood). Moreover a solution exists for all \mathfrak{g} (such a *universal* solution is exhibited in ⁴), but the solution is not unique; moreover, we have shown in ¹⁴ that if we require $\phi_j^i = \delta_j^i +$ higher order terms, then the choice of such a solution ϕ_j^i is equivalent to any among some other data of interest (some of the equivalences known before):

- “ordering prescription”^{5,15}
 - realization of the enveloping algebra in a semi-completed Weyl algebra^{4,14} of the form $\hat{x}_i^\phi = \sum_j x_j \phi_j^i$;
 - a homomorphism of Lie algebras $\phi : \mathfrak{g} \rightarrow Der(\hat{S}(\mathfrak{g}^*))$ (then $\phi_i^j = \phi(-\hat{x}_i)(\partial^j)$).
- It extends to a Hopf action also denoted $\phi : U(\mathfrak{g}) \rightarrow End(\hat{S}(\mathfrak{g}^*))$.
- a deformed Leibniz rule showing how ∂^i acts on a product $\hat{u}\hat{v}$ of $\hat{u}, \hat{v} \in U(\mathfrak{g})$;
 - (topological) coproduct $\Delta : S(\mathfrak{g}^*) \rightarrow \hat{S}(\mathfrak{g}^*) \hat{\otimes} \hat{S}(\mathfrak{g}^*)$;
 - prescription for multiplying certain formal exponentials of a noncommutative argument^{16,13,5} (not shown in ¹⁴);
 - a choice of the star products (belonging to a specific class of star products);
 - a coalgebra isomorphism $\xi : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ such that $\xi|_{\mathfrak{g}} = id_{\mathfrak{g}}$;

For the purpose of the proofs we sketch below some of the relations among the above data, for more see ¹⁴ and Sec. 2.

1.3. (Hopf actions and smash products) Recall that a left action $\triangleright : H \otimes A \rightarrow A$ of a Hopf algebra H on an algebra A is a **Hopf action** if it is satisfying the condition $h \triangleright (a \cdot b) = \sum (h_{(1)} \triangleright a) \cdot (h_{(2)} \triangleright b)$, where we used the Sweedler notation $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$; we also say that A is a left H -module algebra. In that case, one defines the smash product algebra (or crossed product) $A \# H$ as the tensor product $A \otimes H$ with the associative multiplication given by

$$(a \otimes h)(b \otimes g) = \sum (ah_{(1)} \triangleright b) \otimes (h_{(2)}g).$$

1.4. (Heisenberg double) The input for the Heisenberg double^{6,10,18} construction is a pair of Hopf algebras H, H' in a bilinear pairing $\langle \cdot, \cdot \rangle : H \otimes H' \rightarrow \mathbf{k}$ which is Hopf, i.e. with the product on pairings on the tensor square, the coproduct and the product are dual in the sense $\langle \Delta_H(a), b \otimes c \rangle = \langle a, b \cdot c \rangle$, $\langle a \otimes a', \Delta_{H'}b \rangle = \langle a \cdot a', b \rangle$ and similarly for the unit and counit. In our case $H = U(\mathfrak{g})$ and the role of H' is played by the algebraic linear dual $U^*(\mathfrak{g}) = \text{Hom}_{\mathbf{k}}(U(\mathfrak{g}), \mathbf{k})$ which is a *topological Hopf algebra*, i.e. the coproduct of the generators may result in infinitely many summands from the tensor square, amounting to the need for some completion of $H \otimes H'$. Similar to the Drinfel'd double, Heisenberg double is the algebra whose underlying space is (a completion of) $H \otimes H'$, but unlike Drinfel'd double it does not have a Hopf algebra structure itself. One defines the **coregular action** of H' on H given by $h' \triangleright h = \sum h_{(1)} \langle h_{(2)}, h' \rangle$ where $\Delta_H(h) = \sum h_{(1)} \otimes h_{(2)}$; as we required that the pairing is Hopf pairing, this action of H' on H is automatically a Hopf action (cf. **1.3**), hence we can form the corresponding smash product algebra $H \# H'$, the Heisenberg double of H (better, of the data $(H, H', \langle \cdot, \cdot \rangle)$).

1.5. (Sketch of the proof of the main result)

We want to exhibit the isomorphism between the phase space algebra with ϕ -deformed derivatives, and the Heisenberg double of $U(\mathfrak{g})$. This comprises four steps/isomorphisms, the first two of which were effectively done in our earlier work¹⁴, and the remaining step is the focus of this paper.

I By the definition, the phase space algebra with the ϕ -deformed derivatives is generated by $\hat{x}_1, \dots, \hat{x}_n$ in $U(\mathfrak{g})$ and the mutually commuting formal power series in $\partial^1, \dots, \partial^n$ with commutation relation (2). In ¹⁴, we have shown that it is isomorphic to the smash product $U(\mathfrak{g}) \#_{\phi} \hat{S}(\mathfrak{g}^*)$.

II By ¹⁴, ϕ induces an isomorphism of coalgebras denoted $\xi_{\phi} : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$. The transpose of ξ_{ϕ} is an isomorphism of topological algebras $\xi_{\phi}^T : S(\mathfrak{g})^* \rightarrow U(\mathfrak{g})^*$ which one composes with the isomorphism $\hat{S}(\mathfrak{g}^*) \cong S(\mathfrak{g})^*$. Therefore the algebra isomorphism $U(\mathfrak{g}) \#_{\phi} \hat{S}(\mathfrak{g}^*) \cong U(\mathfrak{g}) \# U(\mathfrak{g})^*$ where the action used for the smash product also transfers to the right hand side. This isomorphism is also exhibited in ¹⁴. Notice that the smash product on the right hand side is *not* yet the Heisenberg double as the action used is the action of $U(\mathfrak{g})$ on $U(\mathfrak{g})^*$ and not the conversely.

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IIB Note that the isomorphism $\hat{S}(\mathfrak{g}^*) \cong U(\mathfrak{g})^*$ obtained via ξ_ϕ^T (cf. II) and the identification $\hat{S}(\mathfrak{g}^*) \cong S(\mathfrak{g})^*$ induces also a nondegenerate Hopf pairing between $\hat{S}(\mathfrak{g}^*)$ and $U(\mathfrak{g})$. For this pairing we find several descriptions (5),(7) which are used below to describe the Heisenberg double of $U(\mathfrak{g})$.

III The smash product algebra depends on the action used in its definition. The smash product $U(\mathfrak{g})\#_\phi\hat{S}(\mathfrak{g}^*)$ in II is derived from the action \triangleright of the Hopf algebra $U(\mathfrak{g})$ on $\hat{S}(\mathfrak{g}^*)$. We relate this action with the “black” action \blacktriangleright (see 2.4) of the topological algebra $\hat{S}(\mathfrak{g}^*) \cong U(\mathfrak{g})^*$, and show that the two resulting smash products (one from action of $U(\mathfrak{g})$ on $S(\mathfrak{g})$ and another from the black action of $U(\mathfrak{g}^*)$ on $U(\mathfrak{g})$) are isomorphic as abstract algebras. The action \blacktriangleright is a Hopf action with respect to the topological coproduct on $U(\mathfrak{g})^*$ or equivalently the ϕ -deformed coproduct on $\hat{S}(\mathfrak{g}^*)$.

IV We show in **3.5** that the black action \blacktriangleright is precisely the coregular action, i.e. the unique action satisfying $P \blacktriangleright \hat{u} = \sum \hat{u}_{(1)} \langle \hat{u}_{(2)}, P \rangle_\phi$ where $\langle \cdot, \cdot \rangle_\phi$ is the Hopf pairing with the dual topological Hopf algebra (with the dual represented in a specific way). The coregular action is used in the definition of the Heisenberg double, completing the identification $U(\mathfrak{g})\#_\phi\hat{S}(\mathfrak{g}^*) \cong U(\mathfrak{g})\#U(\mathfrak{g})^*$ where for the smash products, on the left hand side one uses the $U(\mathfrak{g})$ action by \triangleright , and on the right hand side the coregular $U(\mathfrak{g})^*$ -action.

2. More on deformed derivatives

More familiarity with the structure involved in the method of deformed derivatives is needed later to exhibit its relation to the Heisenberg double. For the users of our results we also sketch the connection to star products.

2.1. (Star product perspective) Lie algebra type noncommutative spaces are simply the deformation quantizations of the linear Poisson structure; given structure constants C_{ij}^k linear in a deformation parameter the enveloping algebras of the Lie algebra \mathfrak{g} given in a base by $[\hat{x}_i, \hat{x}_j] = C_{ij}^k \hat{x}_k$ is viewed as a deformation of the polynomial (symmetric) algebra $S(\mathfrak{g})$ generated by commuting x_1, \dots, x_n . Given any linear isomorphism $\xi : S(\mathfrak{g}) \xrightarrow{\cong} U(\mathfrak{g})$ we transfer the noncommutative product on $U(\mathfrak{g})$ to a \star -product on $S(\mathfrak{g})$, defined by $f \star g = \xi^{-1}(\xi(f) \cdot \xi(g))$. There are many isomorphisms which may play role of ξ , but in order to introduce either the ϕ -deformed derivatives like in ^{1,5,15,14}, or to make the correspondence with the Heisenberg double construction, we need to restrict to ξ which are *coalgebra isomorphisms*; we also require a “small deformation condition” that ξ is the identity on the constant and linear parts, i.e. on $\mathfrak{k} \oplus \mathfrak{g} \subset S(\mathfrak{g})$. Our restriction to coalgebra isomorphisms, singles out a distinguished class of star products quantizing the linear Poisson structure. Kathotia⁷ compares the Kontsevich star product⁸ for linear Poisson structures to the PBW-product which corresponds to the case where ξ is the standard symmetrization (coexponential) map (cf. ⁴, especially Chapter 10); Kontsevich star product is not in our class, although it is equivalent to the PBW

product, which is in our class.

2.2. (Some connections between the basic data) Coalgebra isomorphism $\xi : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ induces a transpose map $\xi^T : U^*(\mathfrak{g}) \rightarrow S^*(\mathfrak{g})$, which is consequently an algebra isomorphism. There is an isomorphism $S^*(\mathfrak{g}) \cong \hat{S}(\mathfrak{g}^*)$ where $\hat{S}(\mathfrak{g}^*)$ denotes a completed symmetric algebra on the dual; the isomorphism depends on a normalization of a pairing between $\hat{S}(\mathfrak{g}^*)$ and $S(\mathfrak{g})$ (cf. ⁴, **10.4**, **10.5**). the functionals in $S^*(\mathfrak{g}) \cong \hat{S}(\mathfrak{g}^*)$ can be identified with the infinite order differential operators with constant coefficients: a differential operator applied to a polynomial in $S(\mathfrak{g})$ and then evaluated at 0, defines a differential operator. If the dual generators of $\mathfrak{g}^* \subset \hat{S}(\mathfrak{g}^*)$ corresponding to the basis x_1, \dots, x_n are denoted as the partial derivatives ∂^i , this rule and identification explains the choice of normalization in ⁴, Sec. **10**. The topological coproduct on $U^*(\mathfrak{g})$ which is the algebraic transpose to the product on $U(\mathfrak{g})$, is (for ξ being the symmetrization map) written as a formal differential operators in $\hat{S}(\mathfrak{g}^*)$ in ¹⁷, where the generalizations for Lie bialgebras are considered. In ¹⁴ we have shown that this deformed coproduct is the same as a coproduct obtained by using Leibniz rules defined in terms of the deformed commutation relations; and in the case of symmetric ordering we have exhibited¹⁴ a Feynman-like diagram expansion summing to what is essentially a Fourier-transformed form of the BCH series.

2.3. As shown in detail in ¹⁴, the coalgebra isomorphism $\xi : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ tautological on $\mathfrak{k} \oplus \mathfrak{g}$ as above, is equivalent to any of several other data listed in the introduction, including the ϕ -data described as follows. The star product $x_i \star f$ is always of the form $\sum_j x_j \phi_j^i(\partial^1, \dots, \partial^n)(f)$ where $(\phi_j^i)_{i,j=1,\dots,n}$ is a matrix of elements in $\hat{S}(\mathfrak{g}^*)$ (formal power series in dual variables $\partial^1, \dots, \partial^n$) satisfying a formal set of differential equations (⁴ ch. 4) equivalent to the statement that the formula $\phi(-\hat{x}_i)(\partial^j) = \phi_j^i$ defines a Lie algebra morphism $\phi : \mathfrak{g} \rightarrow \text{Der}(\hat{S}(\mathfrak{g}^*))$.

The correspondence $\hat{x}_i \mapsto \hat{x}_i^\phi = \sum_j x_j \phi_j^i$ extends to an injective morphism of associative algebras $()^\phi : U(\mathfrak{g}) \rightarrow \hat{A}_{n,\mathfrak{k}}$ where $\hat{A}_{n,\mathfrak{k}}$ is the Weyl algebra of differential operators with polynomial coefficients, completed by the degree of the differential operator (hence we allow formal power series in ∂^i -s but not in x_j -s). This (semi)completed Weyl algebra has the standard Fock representation on $S(\mathfrak{g})$. The Lie algebra homomorphism ϕ extends multiplicatively to a unique homomorphism $U(\mathfrak{g}) \rightarrow \text{End}(\hat{S}(\mathfrak{g}^*))$ (also denoted ϕ), which is a Hopf action (cf. **1.3**). Thus we can form a smash product algebra $A_{\mathfrak{g},\phi} = U(\mathfrak{g}) \# \hat{S}(\mathfrak{g}^*)$, the semicompleted n -th Weyl algebra ($\hat{A}_{n,\mathfrak{k}}$ is the special case of this construction for an abelian Lie algebra). The rule $\hat{x}_i \mapsto \hat{x}_i^\phi, \partial^j \mapsto \partial^j$ extends to a unique homomorphism $A_{\mathfrak{g},\phi} \rightarrow \hat{A}_{n,\mathfrak{k}}$; one easily shows that it is an isomorphism.

2.4. (The action later used for Heisenberg double) Not only $U(\mathfrak{g})$ acts by Hopf action on $\hat{S}(\mathfrak{g}^*)$ (this action was used in the construction of $A_{\mathfrak{g},\phi}$), but also conversely $\hat{S}(\mathfrak{g}^*)$ as a topological Hopf algebra acts on $U(\mathfrak{g})$. The *latter* action \blacktriangleright is in the Main Theorem below identified with the smash product action of the Heisenberg

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double! To define the latter action, $U(\mathfrak{g})$ is embedded as a subalgebra $U(\mathfrak{g})\sharp\mathbf{k} \hookrightarrow A_{\mathfrak{g},\phi}$; and similarly for $\hat{S}(\mathfrak{g}^*)$. The action $a \otimes u \mapsto a \blacktriangleright \hat{u}$, $A_{\mathfrak{g},\phi} \otimes U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ is defined by multiplying within $A_{\mathfrak{g},\phi}$ and then projecting by evaluating the second tensor factor in $A_{\mathfrak{g},\phi} = U(\mathfrak{g})\sharp\hat{S}(\mathfrak{g}^*)$ (as a differential operator) at 1. Thus $U(\mathfrak{g})$ is an $A_{\mathfrak{g},\phi}$ -module, the deformed Fock space where $1_{U(\mathfrak{g})}$ is the ϕ -deformed vacuum. It can be shown¹⁴ that the coalgebra isomorphism $\xi : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ can be computed by composing $S(\mathfrak{g}) \hookrightarrow \hat{A}_{n,\mathbf{k}} \cong A_{\mathfrak{g},\phi} \xrightarrow{\blacktriangleright^{1_{U(\mathfrak{g})}}} U(\mathfrak{g})$.

2.5. (Deformed coproduct) If we define, for $P \in \hat{S}(\mathfrak{g}^*) \hookrightarrow A_{\mathfrak{g},\phi}$, the linear operator $\hat{P} : U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ by $\hat{P}(\hat{u}) = P \blacktriangleright \hat{u}$ or (equivalently, according to¹⁴) by $\hat{P}(\xi(f)) = \xi(P(f))$, then the Leibniz rule holds: $\sum \hat{P}_{(1)}(\hat{u}) \cdot_{U(\mathfrak{g})} \hat{P}_{(2)}(\hat{v}) = P(\hat{u} \cdot_{U(\mathfrak{g})} \hat{v})$ for a unique (ϕ -dependent) deformed coproduct $P \mapsto \Delta(P) = \sum P_{(1)} \otimes P_{(2)}$ on $\hat{S}(\mathfrak{g}^*)$ (with the tensor product allowing infinitely many terms), cf. **4.2**.

3. Relating Heisenberg double to the ϕ -deformed derivatives

3.1. Lemma. *The following nonsymmetric formula for $\Delta(\partial^\mu)$ holds:*

$$\Delta(\partial^\mu) = 1 \otimes \partial^\mu + \partial^\alpha \otimes [\partial^\mu, \hat{x}_\alpha] + \frac{1}{2!} \partial^{\alpha_1} \partial^{\alpha_2} \otimes [[\partial^\mu, \hat{x}_{\alpha_1}], \hat{x}_{\alpha_2}] + \dots \quad (3)$$

The sum has only finitely many terms when applied to an element in $U(\mathfrak{g}) \otimes U(\mathfrak{g})$. Proof is by induction, see¹⁴.

3.2. Lemma. *If $\hat{a} = \sum_{\alpha=1}^n a^\alpha \hat{x}_\alpha$ and $\hat{f} \in U(\mathfrak{g})$ then*

$$\hat{\partial}^\mu(\hat{a}^p \hat{f}) = \sum_{k=0}^{p-1} \binom{p}{k} a^{\alpha_1} a^{\alpha_2} \dots a^{\alpha_k} \hat{a}^{p-k} [[\hat{\partial}^\mu, \hat{x}_{\alpha_1}], \dots, \hat{x}_{\alpha_k}](\hat{f}) \quad (4)$$

Proof. This is a tautology for $p = 0$. Suppose it holds for all p up to some p_0 , and for all \hat{f} . Then set $\hat{g} = \hat{a}\hat{f} = a^\alpha \hat{x}_\alpha$. Then $\hat{\partial}^\mu(\hat{a}^{p_0+1}\hat{f}) = \hat{\partial}^\mu(\hat{a}^{p_0}\hat{g})$ and we can apply (4) to $\hat{\partial}^\mu(\hat{a}^{p_0}\hat{g})$. Now

$$\begin{aligned} [[[\hat{\partial}^\mu, \hat{x}_{\alpha_1}], \dots], \hat{x}_{\alpha_k}](\hat{g}) &= a^{\alpha_k} [[[\hat{\partial}^\mu, \hat{x}_{\alpha_1}], \dots], \hat{x}_{\alpha_k}](\hat{x}_{\alpha_{k+1}}\hat{g}) \\ &= \hat{a}[[[\hat{\partial}^\mu, \hat{x}_{\alpha_1}], \dots], \hat{x}_{\alpha_k}](\hat{f}) + \\ &\quad + a^{\alpha_{k+1}} [[[\hat{\partial}^\mu, \hat{x}_{\alpha_1}], \dots], \hat{x}_{\alpha_k}](\hat{f}). \end{aligned}$$

Collecting the terms and the Pascal triangle identity complete the induction step.

3.2.1. Remark. It is interesting that this lemma was needed and proved in¹⁴ related to certain Feynman diagram type expansion calculation leading to an exact summation result, whereas it will be seen here rather as a step toward and a special case of a formula showing the condition that certain secondary action in the ϕ -deformed derivatives picture (the black action) is precisely the coregular action needed to define the Heisenberg double.

3.3. Theorem. *Given a left Hopf action $\phi : U(\mathfrak{g}) \rightarrow \text{End}(\hat{S}(\mathfrak{g}^*))$, with $\phi(-\hat{x}_i)(\partial^j) = \delta_j^i + O(\partial)$, there is a Hopf pairing $\langle \cdot, \cdot \rangle_\phi : U(\mathfrak{g}) \otimes \hat{S}(\mathfrak{g}^*) \rightarrow \mathbf{k}$ given by*

$$\langle \hat{u}, P \rangle_\phi = \phi(S_{U(\mathfrak{g})}\hat{u})(P)|0 \equiv \phi(S_{U(\mathfrak{g})}\hat{u})(P)(1_{S(\mathfrak{g})}) \quad (5)$$

where $\hat{u} \in U(\mathfrak{g})$, $P \in \hat{S}(\mathfrak{g}^*)$, and $S_{U(\mathfrak{g})}$ is the antipode antiautomorphism of $U(\mathfrak{g})$, and where $\hat{S}(\mathfrak{g}^*)$ is considered a topological Hopf algebra with respect to the ϕ -deformed coproduct.

Proof. Clearly the pairing is well defined; the antipode comes because we use left Hopf actions. The product of differential operators with constant coefficients evaluated at 1 equals the product of their evaluations at 1. Therefore the fact that ϕ is Hopf action implies $\langle \hat{u}, PQ \rangle_\phi = \langle \Delta \hat{u}, P \otimes Q \rangle_\phi$. It is less obvious to verify the other duality: of ϕ -deformed coproduct and the multiplication on $U(\mathfrak{g})$. It is sufficient to show that one has

$$\langle \hat{x}_\alpha \hat{u}, \partial^\mu \rangle_\phi = \langle \hat{x}_\alpha \otimes \hat{u}, \Delta \partial^\mu \rangle_\phi. \quad (6)$$

for all α and all \hat{u} in $U(\mathfrak{g})$. Indeed, extending to $\prod_{i=1}^k x_{\alpha_i} \hat{u}$ for all $(\alpha_1, \dots, \alpha_k)$ can be done by induction on k , using the coassociativity of the coproduct and associativity of the product. Once it is true for any product $\hat{v} \hat{u}$ in the left argument, it is an easy general nonsense, using the already known duality for $\Delta_{U(\mathfrak{g})}$, to extend the property to products of ∂ -s by induction using the following calculation for the induction step

$$\begin{aligned} \langle \hat{v} \hat{u}, P_1 P_2 \rangle_\phi &= \langle \sum \hat{v}_{(1)} \hat{u}_{(1)} \otimes \hat{v}_{(2)} \hat{u}_{(2)}, P_1 \otimes P_2 \rangle_\phi \\ &= \sum \langle \hat{v}_{(1)} \otimes \hat{u}_{(1)}, \Delta(P_1) \rangle_\phi \langle \hat{v}_{(2)} \otimes \hat{u}_{(2)}, \Delta(P_2) \rangle_\phi \\ &= \sum \langle \hat{v}_{(1)} \otimes \hat{u}_{(1)} \otimes \hat{v}_{(2)} \otimes \hat{u}_{(2)}, \Delta(P_1) \otimes \Delta(P_2) \rangle_\phi \\ &= \sum \langle \hat{v} \otimes \hat{u}, \Delta(P_1 P_2) \rangle_\phi \end{aligned}$$

Let us now calculate (6) using the nonsymmetric formula (3) for the ϕ -coproduct. All terms readily give zero in first factor unless the first factor is degree 1 in ∂ -s. Thus we effectively need to show

$$\sum_{\beta} \langle x_\alpha, \partial^\beta \rangle_\phi \otimes \langle \hat{u}, [\partial^\mu, \hat{x}_\beta] \rangle_\phi = \langle x_\alpha \hat{u}, \partial^\mu \rangle_\phi.$$

The left-hand side is $\sum_{\beta} \phi(-\hat{x}_\alpha)(\partial^\beta) \phi(S_{U(\mathfrak{g})}\hat{u}^{\text{op}})(\phi(-\hat{x}_\beta)(\partial^\mu))|0 = \sum_{\beta} \phi(-\hat{x}_\alpha)(\partial^\beta)|0 \phi(S_{U(\mathfrak{g})}(\hat{u}^{\text{op}}\hat{x}_\beta)(\partial^\mu))|0$ and $\phi(-\hat{x}_\alpha)(\partial^\beta)|0 = \delta_\alpha^\beta$ by the assumption on ϕ . Finally, the contraction with the Kronecker delta gives $\phi(S_{U(\mathfrak{g})}(\hat{x}_\alpha \hat{u}^{\text{op}})(\partial^\mu))|0$.

3.4. Proposition. *If $\xi = \xi_\phi : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ is the coalgebra isomorphism corresponding to ϕ and $\xi^T : U(\mathfrak{g})^* \rightarrow S(\mathfrak{g})^* \cong \hat{S}(\mathfrak{g}^*)$ its transpose, then the pairing may be described alternatively by*

$$\langle \hat{u}, P \rangle_\phi = (\xi^T)^{-1}(P)(\hat{u}) = P(\xi_\phi^{-1}(\hat{u})) = \epsilon_{S(\mathfrak{g})}(P(\hat{u}^\phi|0)) = \epsilon_{S(\mathfrak{g})}((\hat{P}(\hat{u}))^\phi|0), \quad (7)$$

where $P(\xi_\phi^{-1}(\hat{u}))$ is the evaluation of $P \in \hat{S}(\mathfrak{g}^*)$ on $\xi_\phi^{-1}(\hat{u}) \in S(\mathfrak{g})$ via the pairing.

Proof. We show $\langle \hat{u}, P \rangle_\phi = \epsilon_{S(\mathfrak{g})}(P(\hat{u}^\phi|0))$. By the previous arguments, it is enough to show that this alternative formula gives the same (and, in particular,

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Hopf) pairing as (5) when $P = \partial^\mu$. This is evident when $\hat{u} = \hat{x}_\nu$ for some ν . Now suppose by induction that (7) holds for \hat{u} . Then

$$\begin{aligned} \epsilon \partial^\mu (\hat{x}_\lambda^\phi \hat{u}^\phi | 0) &= \epsilon \partial^\mu x_\alpha \phi_\lambda^\alpha \hat{u}^\phi | 0) \\ &= \epsilon x_\alpha \partial^\mu \phi_\lambda^\alpha \hat{u}^\phi | 0) + \epsilon \phi_\lambda^\mu \hat{u}^\phi | 0) \\ &= 0 + \epsilon \phi(S_{U(\mathfrak{g})} \hat{u})([\partial^\mu, \hat{x}_\lambda]) \\ &= \epsilon \phi(S_{U(\mathfrak{g})}(\hat{x}_\lambda \hat{u}))(\partial^\mu), \end{aligned}$$

hence it holds for $\hat{x}_\lambda \hat{u}$.

The other equalities in (7) are direct: $(\xi^T)^{-1}(P)(\hat{u}) = P(\xi_\phi^{-1}(\hat{u}))$ by the definition of the transpose operator ξ^T ; then $P(\xi_\phi^{-1}(\hat{u})) = \epsilon(P(\hat{u}^\phi | 0))$ and $\epsilon(P(\hat{u}^\phi | 0)) = \epsilon((\hat{P}(\hat{u}))^\phi | 0)$ by the basic identities $\epsilon(\hat{w} | 0) = \xi^{-1}(\hat{w})$ and $\hat{P} \circ \xi = \xi \circ P$.

3.5. Main Theorem. *The (\mathfrak{g}, ϕ) -twisted Weyl algebra $A_{\mathfrak{g}, \phi}$ is isomorphic to the Heisenberg double of the Hopf algebra $U(\mathfrak{g})$ where the dual topological Hopf algebra is $\hat{S}(\mathfrak{g}^*)$ with respect to the ϕ -deformed coproduct, and with respect to the Hopf pairing given by (5) or, equivalently, (7). In other words, the left action \blacktriangleright used for the second smash product structure satisfies (and is determined by) the formula*

$$P \blacktriangleright \hat{u} = \sum \langle \hat{u}_{(2)}, P \rangle_\phi \hat{u}_{(1)}$$

for all $\hat{u} \in U(\mathfrak{g})$ and $P \in \hat{S}(\mathfrak{g}^*)$.

Consequently, the phase space algebra with the ϕ -deformed derivatives (1.2.2) is isomorphic to the Heisenberg double (and the isomorphism nontrivially depends on ϕ).

Proof. If the identity holds for $P = P_1$ and $P = P_2$ then

$$\begin{aligned} P_1 P_2 \blacktriangleright \hat{u} &= P_1 \blacktriangleright (\sum \langle \hat{u}_{(2)}, P_2 \rangle_\phi \hat{u}_{(1)}) \\ &= \sum \langle \hat{u}_{(3)}, P_2 \rangle_\phi \langle \hat{u}_{(2)}, P_1 \rangle_\phi \hat{u}_{(1)} \\ &= \sum \langle \hat{u}_{(2)}, P_1 P_2 \rangle_\phi \hat{u}_{(1)} \end{aligned}$$

hence it holds for $P = P_1 P_2$. For $P = 1$ it holds trivially, hence it is sufficient to check for $P = \partial^\mu$ and use induction. The identity is linear in $\hat{u} \in U(\mathfrak{g})$, so it is sufficient to prove it for all \hat{u} of the form $\hat{u} = \hat{a}^p = (\sum_{\alpha=1}^n a^\alpha \hat{x}_\alpha)^p$, $p \geq 0$ where $\hat{a} = \sum_\alpha a^\alpha \hat{x}_\alpha$ is arbitrary. In that case, $\Delta(\hat{u}) = \sum_{k=0}^p \binom{p}{k} \hat{a}^{p-k} \otimes \hat{a}^k$ and we need to show

$$\partial^\mu \blacktriangleright \hat{a}^p = \hat{\partial}^\mu(\hat{a}^p) = \sum_{k=0}^p \binom{p}{k} \langle \hat{a}^k, \partial^\mu \rangle_\phi \hat{a}^{n-k}$$

but $\langle \hat{a}^k, \partial^\mu \rangle_\phi$ is by (7) equal to

$$\phi(S_{U(\mathfrak{g})}(\hat{a}^k))(\partial^\mu) = (-1)^k \phi(\hat{a}^k)(\partial^\mu) = (-1)^k [\dots [[\partial^\mu, \hat{a}], \hat{a}], \dots, \hat{a}],$$

what by linearity reduces to (4) for the case $f = 1$. (This shows IV in 1.5 i.e. that the black action \blacktriangleright is identifiable with the coregular action under the isomorphism $\hat{S}(\mathfrak{g}^*) \cong U(\mathfrak{g}^*)$).

While the vector spaces of the two smash products ($U(\mathfrak{g})\sharp_{\phi}\hat{S}(\mathfrak{g})$ and the Heisenberg double) are isomorphic by the definition (they are simply the tensor products with the same factors), we need to show that the multiplication is the same; for this we need to commute the tensor factors. One can easily compute that if $[\partial, \hat{x}] = Q \in \hat{S}(\mathfrak{g}^*)$, then also in the Heisenberg double $[\hat{\partial}, \hat{x}] = \hat{Q}$ for $\partial \in \mathfrak{g}^*$ and $\hat{x} \in \mathfrak{g} \hookrightarrow U(\mathfrak{g})$. Therefore for the generators, the commutation relations in the two smash products agree (this shows III in 1.5), hence the isomorphism of $A_{\mathfrak{g},\phi}$ and the smash product given by the black action, hence the Heisenberg double.

The final sentence in the theorem now follows by I in 1.5, namely we know from our earlier work¹⁴ that the phase space algebra with the deformed derivatives is isomorphic to the smash product $A_{\mathfrak{g},\phi}$. Step II in 1.5 shown in ¹⁴ is used all along in the construction. Notice that the heart of this paper is performing the step IV from 1.5; once we have done it, we have recapitulated earlier prepared steps for I, II and III.

4. Final remarks.

4.1. Though the ϕ -deformed derivatives are not present there, the Reshetikhin's article¹⁷ has implicitly much of the structure from this paper (including issues on dualization of coproducts) implicitly present, including the quantum deformations of enveloping algebras and more general bialgebras.

4.2. The fact that the Leibniz rule for the action of $\hat{S}(\mathfrak{g}^*)$ on $U(\mathfrak{g})$ (for any \mathfrak{g} and ϕ) gives a well-defined coassociative map into the tensor product is not obvious in the deformed derivative picture^{1,3,14,15}; namely it is *a priori* undefined up to a kernel of the multiplication map (add an element in the kernel and the Leibniz rule does not change). But now the Hopf action is well-defined within the Heisenberg double construction and the Heisenberg double as an algebra is identified with $A_{\mathfrak{g},\phi}$ where the deformed Leibniz rule was originally defined. Heisenberg double provides an invariant picture, giving simple "dual" interpretation to the deformed coproduct, while the approach via the ϕ -deformed derivatives and commutators is useful for calculation, as it is exhibited in the physics literature before.

4.3. The differential forms and exterior derivative can also be extended to the same setup^{19,12}.

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