COMBINATORIAL BASES OF FEIGIN-STOYANOVSKY'S TYPE SUBSPACES OF LEVEL 1 STANDARD MODULES FOR $\tilde{\mathfrak{sl}}(\ell+1,\mathbb{C})$

GORAN TRUPČEVIĆ

Abstract. Let $\tilde{\mathfrak{g}}$ be an affine Lie algebra of the type $A_\ell^{(1)}$. Suppose we are given a $\mathbb{Z}$-gradation of the corresponding simple finite-dimensional Lie algebra $\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$; then we also have the induced $\mathbb{Z}$-gradation of the affine Lie algebra

$$\tilde{\mathfrak{g}}=\tilde{\mathfrak{g}}_{-1}\oplus\tilde{\mathfrak{g}}_0\oplus\tilde{\mathfrak{g}}_1.$$

Let $L(\Lambda)$ be a standard module of level 1. Feigin-Stoyanovsky's type subspace $W(\Lambda)$ is the $\tilde{\mathfrak{g}}_1$-submodule of $L(\Lambda)$ generated by the highest-weight vector v_Λ,

$$W(\Lambda)=U(\tilde{\mathfrak{g}}_1)\cdot v_\Lambda\subset L(\Lambda).$$

We find a combinatorial basis of $W(\Lambda)$ given in terms of difference and initial conditions.

1. Introduction

Let $\mathfrak{g}$ be a simple complex Lie algebra, $\mathfrak{h}\subset\mathfrak{g}$ its Cartan subalgebra, R the corresponding root system. Then one has a root decomposition $\mathfrak{g}=\mathfrak{h}+\sum_{\alpha\in R}\mathfrak{g}_\alpha$. Fix root vectors $x_\alpha\in\mathfrak{g}_\alpha$. Let

$$\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1 \tag{1}$$

be a $\mathbb{Z}$-gradation of $\mathfrak{g}$, where $\mathfrak{h}\subset\mathfrak{g}_0$. Denote by $\Gamma\subset R$ the set of roots such that $\mathfrak{g}_1=\sum_{\alpha\in\Gamma}\mathfrak{g}_\alpha$.

The affine Lie algebra associated with $\mathfrak{g}$ is $\tilde{\mathfrak{g}}=\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}c\oplus\mathbb{C}d$, where c is the canonical central element, and d the degree operator. Elements $x_\alpha(n)=x_\alpha\otimes t^n$ are fixed real root vectors. The $\mathbb{Z}$-gradation of $\mathfrak{g}$ induces the $\mathbb{Z}$-gradation of $\tilde{\mathfrak{g}}$:

$$\tilde{\mathfrak{g}}=\tilde{\mathfrak{g}}_{-1}\oplus\tilde{\mathfrak{g}}_0\oplus\tilde{\mathfrak{g}}_1,$$

where $\tilde{\mathfrak{g}}_1=\mathfrak{g}_1\otimes\mathbb{C}[t,t^{-1}]$ is a commutative Lie subalgebra with basis

$$\{x_\gamma(j)\,|\,j\in\mathbb{Z},\gamma\in\Gamma\}.$$

Let $L(\Lambda)$ be a standard $\tilde{\mathfrak{g}}$-module of level $k=\Lambda(c)$, with a fixed highest weight vector v_Λ. A Feigin-Stoyanovsky's type subspace is a $\tilde{\mathfrak{g}}_1$-submodule of $L(\Lambda)$ generated by v_Λ,

$$W(\Lambda)=U(\tilde{\mathfrak{g}}_1)\cdot v_\Lambda\subset L(\Lambda).$$

We find a *monomial basis* of $W(\Lambda)$, i.e. a basis consisting of vectors $x(\pi)v_\Lambda$, where $x(\pi)$ are monomials in basis elements $\{x_\gamma(-j)\,|\,j\in\mathbb{N},\gamma\in\Gamma\}$.

The problem of finding monomial bases is a part of Lepowsky-Wilson's program to study representations of affine Lie algebras by means of vertex operators and to obtain Rogers-Ramanujan-type combinatorial bases of these representations (Lepowsky and Wilson (1984), Lepowsky and Primc (1985), Meurman and Primc (1999)).

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The notion of Feigin-Stoyanovsky's type subspaces is similar to the notion of principal subspaces of standard $\hat{\mathfrak{g}}$-modules, introduced by B. Feigin and A. Stoyanovsky (1994). These subspaces are generated by the affinization of the nilpotent subalgebra $\mathfrak{n}_+$ of $\mathfrak{g}$ coming from the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$; in the case of $\mathfrak{sl}(2, \mathbb{C})$, definitions of principal subspaces and Feigin-Stoyanovsky's type subspaces are equivalent. Feigin and Stoyanovsky described the dual space of the principal subspace for $\mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{sl}(3, \mathbb{C})$ in terms of symmetric polynomial forms satisfying certain conditions, and calculated its character. In the $\mathfrak{sl}(2, \mathbb{C})$-case, they also described the dual in a geometric way, recovering in this way the Rogers-Ramanujan and Gordon identities. Furthermore, by representing the whole standard module $L(\Lambda)$ as an inductive limit of Weyl-group translates of the principal subspace, they constructed a basis of $L(\Lambda)$ consisting of semi-infinite monomials.

Principal subspaces were studied further by G. Georgiev in (1996). He constructed combinatorial bases and calculated characters of principal subspaces for certain representations of $\mathfrak{sl}(\ell+1, \mathbb{C})$. In the proof of linear independence, Georgiev used intertwining operators from Dong and Lepowsky (1993).

Also by using intertwining operators, S. Capparelli, J. Lepowsky and A. Milas in (2003, 2006) obtained Rogers-Ramanujan and Rogers-Selberg recursions for characters of principal subspaces for $\mathfrak{sl}(2, \mathbb{C})$. As a continuation of this program, C. Calinescu obtained systems of recursions for characters of principal subspaces of level 1 standard modules for $\mathfrak{sl}(\ell + 1, \mathbb{C})$ in Calinescu (2008) and of certain higher-level standard modules for $\mathfrak{sl}(3, \mathbb{C})$ in Calinescu (2007). By solving these recursions they also established formulas for the characters of these subspaces. Furthermore, in Calinescu, Lepowsky and Milas (2008a,b) new proofs of presentation theorems for principal subspaces for $\mathfrak{sl}(2, \mathbb{C})$ were provided.

Feigin-Stoyanovsky's type subspace $W(\Lambda)$ was implicitly studied by M. Primc (1994, 2000), who constructed a combinatorial basis of this subspace, and, in a similar way to which it is done in Feigin and Stoyanovsky (1994), obtained a basis of the whole $L(\Lambda)$. This was done in Primc (1994) for $\mathfrak{sl}(\ell + 1, \mathbb{C})$ and a particular choice of gradation (1), and for any dominant integral weight Λ. For any classical simple Lie algebra and any possible gradation (1), combinatorial bases were constructed in Primc (2000), but only for basic modules $L(\Lambda_0)$.

In the particular $\mathfrak{sl}(\ell + 1, \mathbb{C})$ case studied in Primc (1994), the basis of $W(\Lambda)$ is parameterized by combinatorial objects called $(k, \ell + 1)$-admissible configurations. These objects were introduced and further studied by Feigin, Jimbo, Loktev, Miwa and Mukhin (2003) and Feigin, Jimbo, Miwa, Mukhin and Takeyama (2004a,b), where different formulas for the character of $W(\Lambda)$ were obtained. Also, by using combinatorial bases and certain coefficients of intertwining operators, M. Jerković (2009) obtained exact sequences of Feigin-Stoyanovsky's type subspaces at fixed level k, which led to systems of recurrence relations for formal characters of those subspaces.

The hardest part of constructing a combinatorial basis of $W(\Lambda)$ is a proof of linear independence of a reduced spanning set. This was proved in Primc (1994) by using Schur functions, while in Primc (2000) this was proved by using the crystal base character formula of Kang, Kashiwara, Misra, Miwa, Nakashima and Nakayashiki (1992). In Primc (2007) the Capparelli-Lepowsky-Milas' approach via intertwining operators and the description of the basis from Feigin, Jimbo, Loktev, Miwa and Mukhin (2003) were used to give a simpler proof of linear independence of the basis of $W(\Lambda)$ constructed in Primc (1994). It seems that this should be the way to obtain a proof in other cases as well.

In this paper we extend these results to any possible $\mathbb{Z}$-gradation of $\mathfrak{g} = \mathfrak{sl}(\ell + 1, \mathbb{C})$ and all level 1 standard modules. In Trupčević (2009) we further extend this to standard modules of any higher level, obtaining a combinatorial basis parameterized by a certain generalization of $(k, \ell+1)$-admissible configurations.

Let $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ be a basis of the root system R for $\mathfrak{g} = \mathfrak{sl}(\ell+1, \mathbb{C})$, and $\{\omega_1, \dots, \omega_\ell\}$ the corresponding set of fundamental weights. We identify $\mathfrak{h}$ and $\mathfrak{h}^*$ via the invariant bilinear form $\langle x, y\rangle = \operatorname{tr} xy$ and fix a fundamental weight $\omega = \omega_m$. Set

$$\Gamma = \{\gamma \in R \,|\, \langle \gamma, \omega\rangle = 1\} = \{\gamma_{ij} \,|\, i = 1, \dots, m; j = m, \dots, \ell\},$$

where

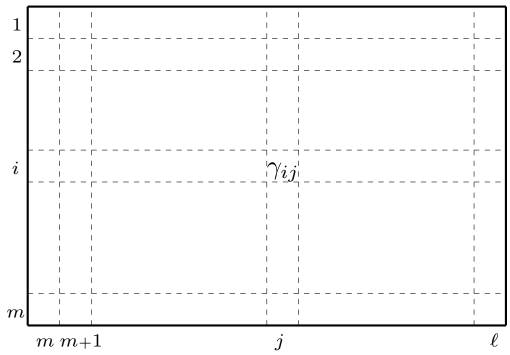
$$\gamma_{ij} = \alpha_i + \cdots + \alpha_m + \cdots + \alpha_j.$$

The $\mathbb{Z}$-gradation of $\mathfrak{g}$ is given by

$$\mathfrak{g}_{\pm 1} = \sum_{\alpha \in \pm\Gamma} \mathfrak{g}_\alpha, \;\; \mathfrak{g}_0 = \mathfrak{h} \oplus \sum_{\langle \alpha, \omega\rangle = 0} \mathfrak{g}_\alpha.$$

The set Γ is called *the set of colors*. For $\gamma \in \Gamma$, we say that a fixed basis element $x_\gamma \in \mathfrak{g}_\gamma$ *is of the color* γ. The set of colors Γ can be pictured as a rectangle with row indices $1, \dots, m$ and column indices $m, \dots, \ell$ (see figure 1).

FIGURE 1. The set of colors Γ



Fix a fundamental weight $\Lambda_i, i = 0, \dots, \ell$ of $\tilde{\mathfrak{g}}$. Let $L(\Lambda_i)$ be the standard module with highest weight Λ_i, and v_i the highest weight vector of $L(\Lambda_i)$.

We find a basis of the Feigin-Stoyanovsky's type subspace $W(\Lambda_i)$ consisting of *monomial vectors*

$$\{x_{\gamma_1}(-n_1) \cdots x_{\gamma_t}(-n_t) v_i \,|\, t \in \mathbb{Z}_+; \gamma_j \in \Gamma, n_j \in \mathbb{N}\}$$

whose *monomial parts*

$$x_{\gamma_1}(-n_1) \cdots x_{\gamma_t}(-n_t) \tag{2}$$

satisfy certain combinatorial conditions, called *difference* and *initial conditions*. We say that the monomial (2) satisfies difference conditions if the colors of its elements of degree $-j$ and $-j-1$ lie on diagonal paths in Γ, as pictured in figure 2.

So, if a monomial (2) has elements of degrees $-j$ and $-j-1$ of colors $\gamma_{r_1 s_1}, \dots, \gamma_{r_t s_t}$ and $\gamma_{r'_1 s'_1}, \dots, \gamma_{r'_{t'} s'_{t'}}$, respectively, then

$$r_1 < r_2 < \cdots < r_t \text{ and } s_1 > s_2 > \cdots > s_t,$$

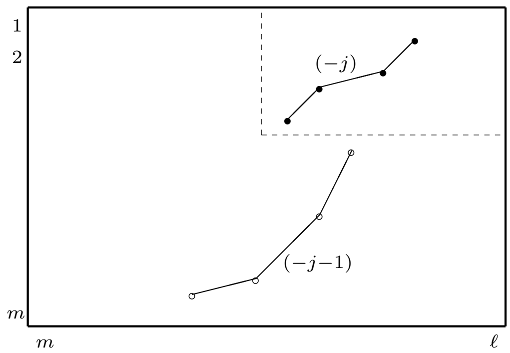
and, similarly,

$$r'_1 < r'_2 < \cdots < r'_{t'} \text{ and } s'_1 > s'_2 > \cdots > s'_{t'}.$$

Moreover,

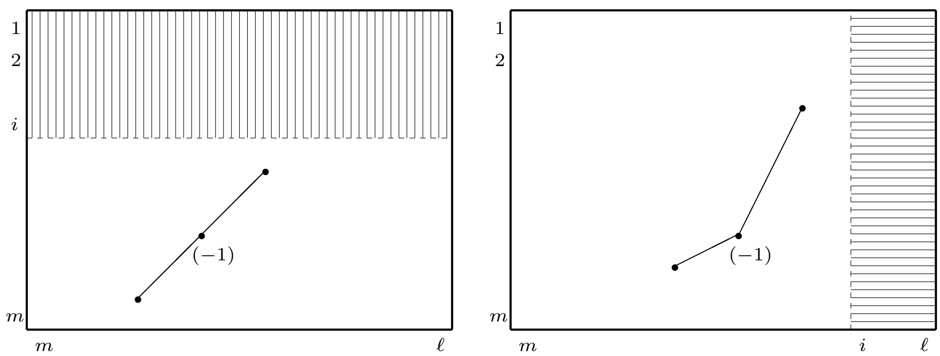
$$r_t < r'_1 \text{ or } s_t > s'_1.$$

Figure 2. Difference conditions



Initial conditions on the monomial (2) require that the diagonal path of colors of elements of degree -1 lie below the i-th row, in the case $1 \leq i \leq m$, or on the left of the i-th column, in the case of $m \leq i \leq \ell$, as pictured in figure 3.

Figure 3. Initial conditions



Difference conditions are obtained by observing relations between fields $x_\gamma(z), \gamma \in \Gamma$ on $L(\Lambda_i)$, while initial conditions follow from the obvious requirement that elements of degree -1 do not annihilate the highest weight vector v_i.

By observing configurations of colors of elements of degrees -1 and -2, one is able to construct coefficients of suitable intertwining operators between standard modules that send some basis elements of one module to basis elements of another module, and annihilate the rest of the basis elements. These operators are then used for the inductive proof of linear independence.

Thus we are able to prove the main result of this paper

Theorem 6 *Let $L(\Lambda_i)$ be a standard module of level 1. Then the set of monomial vectors $x_{\gamma_1}(-n_1)\cdots x_{\gamma_t}(-n_t)v_i$ whose monomial part satisfies difference and initial conditions, is a basis of $W(\Lambda_i)$.*

2. Affine Lie algebras

For $\ell \in \mathbb{N}$, let

$$\mathfrak{g} = \mathfrak{s}l(\ell + 1, \mathbb{C}),$$

a simple Lie algebra of the type A_ℓ. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{g}$ and R the corresponding root system. Fix a basis $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ of R. We have the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Denote by R_+ and R_- sets of positive and negative roots, and by θ the maximal root. Let $\langle x, y\rangle = \operatorname{tr} xy$ be a normalized invariant bilinear form on $\mathfrak{g}$; via $\langle \cdot, \cdot \rangle$ we have an identification $\nu : \mathfrak{h} \to \mathfrak{h}^*$. For each root α fix a root vector $x_\alpha \in \mathfrak{g}_\alpha$.

Let $\{\omega_1, \dots, \omega_\ell\}$ be the set of fundamental weights of $\mathfrak{g}$, $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$, $i, j = 1, \dots, \ell$. Denote by $Q = \sum_{i=1}^{\ell} \mathbb{Z}\alpha_i$ the root lattice, and by $P = \sum_{i=1}^{\ell} \mathbb{Z}\omega_i$ the weight lattice of $\mathfrak{g}$.

Denote by $\tilde{\mathfrak{g}}$ the associated affine Lie algebra

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

(cf. Kac (1990)). Set $x(j) = x \otimes t^j$ for $x \in \mathfrak{g}, j \in \mathbb{Z}$. Commutation relations are given by

$$\begin{aligned}
[c, \tilde{\mathfrak{g}}] &= 0, \\
[d, x(j)] &= jx(j), \\
[x(i), y(j)] &= [x, y](i + j) + i\langle x, y\rangle \delta_{i+j,0} c.
\end{aligned}$$

Set $\mathfrak{h}^e = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$, $\tilde{\mathfrak{n}}_\pm = \mathfrak{g} \otimes t^{\pm 1}\mathbb{C}[t^{\pm 1}] \oplus \mathfrak{n}_\pm$. Then we also have the triangular decomposition $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h}^e \oplus \tilde{\mathfrak{n}}_+$.

Let $\hat{\Pi} = \{\alpha_0, \alpha_1, \dots, \alpha_\ell\} \subset (\mathfrak{h}^e)^*$ be the set of simple roots of $\tilde{\mathfrak{g}}$. The usual extensions of bilinear forms $\langle \cdot, \cdot \rangle$ onto $\mathfrak{h}^e$ and $(\mathfrak{h}^e)^*$ are denoted by the same symbols (we take $\langle c, d\rangle = 1$). Define fundamental weights $\Lambda_i \in (\mathfrak{h}^e)^*$ by $\langle \Lambda_i, \alpha_j \rangle = \delta_{ij}$ and $\Lambda_i(d) = 0$, $i, j = 0, \dots, \ell$.

Let V be a highest weight module for the affine Lie algebra $\tilde{\mathfrak{g}}$ with the highest weight $\Lambda \in (\mathfrak{h}^e)^*$. Then V is generated by a highest weight vector v_Λ such that

$$\begin{aligned}
h \cdot v_\Lambda &= \Lambda(h) v_\Lambda, \quad \text{for } h \in \mathfrak{h}^e, \\
x \cdot v_\Lambda &= 0, \quad \text{for } x \in \tilde{\mathfrak{n}}_+.
\end{aligned}$$

The module V is a direct sum of weight subspaces $V_\mu = \{v \in V \mid h \cdot v = \mu(h)v \text{ for } h \in \mathfrak{h}^e\}$, $\mu \in (\mathfrak{h}^e)^*$.

Standard (i.e. integrable highest weight) $\tilde{\mathfrak{g}}$-module $L(\Lambda)$ is an irreducible highest weight module, with the highest weight Λ being dominant integral, i.e.

$$\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + \cdots + k_\ell\Lambda_\ell,$$

where $k_i \in \mathbb{Z}_+$, $i = 0, \dots, \ell$. The central element c acts on $L(\Lambda)$ as multiplication by the scalar

$$k = \Lambda(c) = k_0 + k_1 + \cdots + k_\ell,$$

which is called the level of the module $L(\Lambda)$.

3. Feigin-Stoyanovsky's type subspace

Vector $v \in \mathfrak{h}$ is said to be *cominuscule* if

$$\{\alpha(v) \mid \alpha \in R\} \in \{-1, 0, 1\}.$$

Similarly, weight $\omega \in P$ is said to be *minuscule* if

$$\{\langle \omega, \alpha\rangle \mid \alpha \in R\} \in \{-1, 0, 1\}.$$

One immediately sees that a dominant integral weight $\omega \in P^+$ is minuscule if and only if $\langle \omega, \theta\rangle = 1$. So, there exists a finite number of minuscule weights. Furthermore, a vector $v \in \mathfrak{h}$ is cominuscule if and only if it is dual to some minuscule fundamental weight ω, in the sense that $v = \nu^{-1}(\omega)$, for some choice of positive roots.

Fix a cominuscule vector $v \in \mathfrak{h}$. In the case of $\mathfrak{g} = \mathfrak{s}l(\ell+1, \mathbb{C})$, all fundamental weights are minuscule. Then we can assume that the cominuscule vector v is dual to a fundamental weight

$$\omega = \omega_m,$$

for some $m \in \{1, \ldots, \ell\}$. Set

$$\Gamma = \{ \alpha \in R \mid \alpha(v) = 1\} = \{ \alpha \in R \mid \langle \omega, \alpha \rangle = 1\}.$$

Then we have the induced $\mathbb{Z}$-gradation of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1, \tag{3}$$

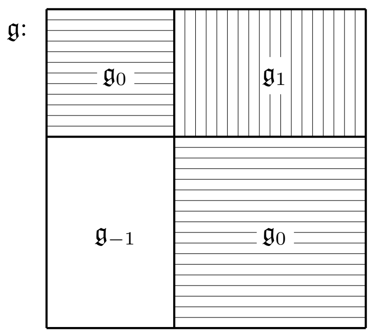
where

$$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{h} \oplus \sum_{\alpha(v)=0} \mathfrak{g}_\alpha \\ \mathfrak{g}_{\pm 1} &= \sum_{\alpha \in \pm\Gamma} \mathfrak{g}_\alpha. \end{aligned}$$

Subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are commutative, and $\mathfrak{g}_0$ acts on them by adjoint action. The subalgebra $\mathfrak{g}_0$ is reductive with semisimple part $\mathfrak{l}_0 = [\mathfrak{g}_0, \mathfrak{g}_0]$ of the type $A_{m-1} \times A_{\ell-m}$; as a root basis one can take $\{\alpha_1, \ldots, \alpha_{m-1}\} \cup \{\alpha_{m+1}, \ldots, \alpha_\ell\}$, and the center is equal to $\mathbb{C}v$.

We illustrate decomposition (3) on the picture 4, which corresponds to the usual realization of $\mathfrak{g}$ as matrices of trace 0. In this case the subalgebra $\mathfrak{g}_0$ consists of block-diagonal matrices, while $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ consist of matrices with non-zero entries only in the upper right or lower-left block, respectively.

FIGURE 4. $\mathbb{Z}$-gradation of $\mathfrak{g}$



The set Γ is called the *set of colors*; it is equal to

$$\Gamma = \{\gamma_{ij} \mid i = 1, \ldots, m; j = m, \ldots, \ell\}$$

where

$$\gamma_{ij} = \alpha_i + \cdots + \alpha_m + \cdots + \alpha_j \tag{4}$$

(see figure 1). The maximal root θ is equal to $\gamma_{1\ell}$.

Similarly, one also has the induced $\mathbb{Z}$-gradation of the affine Lie algebra $\tilde{\mathfrak{g}}$:

$$\begin{aligned} \tilde{\mathfrak{g}}_0 &= \mathfrak{g}_0 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d, \\ \tilde{\mathfrak{g}}_{\pm 1} &= \mathfrak{g}_{\pm 1} \otimes \mathbb{C}[t, t^{-1}], \\ \tilde{\mathfrak{g}} &= \tilde{\mathfrak{g}}_{-1} + \tilde{\mathfrak{g}}_0 + \tilde{\mathfrak{g}}_1. \end{aligned}$$

As above, $\tilde{\mathfrak{g}}_{-1}$ and $\tilde{\mathfrak{g}}_1$ are commutative subalgebras, and $\tilde{\mathfrak{g}}_1$ is a $\tilde{\mathfrak{g}}_0$-module.

For a dominant integral weight Λ, we define a *Feigin-Stoyanovsky's type subspace*

$$W(\Lambda) = U(\tilde{\mathfrak{g}}_1) \cdot v_\Lambda \subset L(\Lambda).$$

Our objective is to find a combinatorial basis of $W(\Lambda)$. Set

$$\tilde{\mathfrak{g}}_1^+ = \tilde{\mathfrak{g}}_1 \cap \tilde{\mathfrak{n}}_+, \ \tilde{\mathfrak{g}}_1^- = \tilde{\mathfrak{g}}_1 \cap \tilde{\mathfrak{n}}_-.$$

Then we have

$$W(\Lambda) = U(\tilde{\mathfrak{g}}_1^-) \cdot v_\Lambda.$$

By Poincaré-Birkhoff-Witt theorem, we have a spanning set of $W(\Lambda)$ consisting of monomial vectors

$$\{x_{\gamma_1}(-n_1)x_{\gamma_2}(-n_2)\cdots x_{\gamma_r}(-n_r)v_\Lambda \,|\, r \in \mathbb{Z}_+; \gamma_j \in \Gamma, n_j \in \mathbb{N}\}. \tag{5}$$

Elements of the spanning set (5) can be identified with monomials from $U(\tilde{\mathfrak{g}}_1) = S(\tilde{\mathfrak{g}}_1)$. With this in mind, we often refer to elements $x_\gamma(-j)$, $\gamma \in \Gamma, j \in \mathbb{Z}$ as *variables*, *elements* or *factors* of monomials.

We can also identify monomials from $S(\tilde{\mathfrak{g}}_1)$ with *colored partitions.* From the beginnings of the representation-theoretic approach to Rogers-Ramanujan identities, combinatorial basis of certain representations were parameterized by partitions satisfying certain conditions (cf. Lepowsky and Wilson (1984), Lepowsky and Primc (1985)). Let $\pi : \{x_\gamma(-j) \mid \gamma \in \Gamma, j \in \mathbb{Z}\} \to \mathbb{Z}_+$ be a colored partition (cf. Primc (1994), section 3). The corresponding monomial $x(\pi) \in S(\tilde{\mathfrak{g}}_1)$ is

$$x(\pi) = x_{\gamma_1}(-j_1)^{\pi(x_{\gamma_1}(-j_1))} \cdots x_{\gamma_t}(-j_t)^{\pi(x_{\gamma_t}(-j_t))}.$$

From this identification we take the notation $x(\pi)$ for the monomials from $S(\tilde{\mathfrak{g}}_1)$. Sometimes, it will be convenient to define monomials by using this identification. Also, our combinatorial conditions for the basis elements can be written in terms of the exponents $\pi(x_\gamma(-j))$, which gives a parametrization of the basis by a generalization of the notion of $(k, \ell+1)$-admissible configurations from Feigin, Jimbo, Loktev, Miwa and Mukhin (2003). This will be useful in the higher-level case (cf. Trupčević (2009)).

4. Order on the set of monomials

We introduce a linear order on the set of monomials.

On the weight and root lattice, we have the usual partial order: for $\mu, \nu \in P$ set $\mu \prec \nu$ if $\mu - \nu$ is a nonnegative integral linear combination of simple roots α_i, $i = 1, \dots, \ell$.

Next, we define a linear order $<$ on the set of colors Γ which is an extension of the order $\prec$. For elements of Γ, $\gamma_{i'j'} \prec \gamma_{ij}$ is equivalent to saying that $i' \geq i$ and $j' \leq j$. The order $<$ on Γ is defined in the following way:

$$\gamma_{i'j'} < \gamma_{ij} \quad \text{if} \quad \begin{cases} i' > i \\ i' = i, \ j' < j. \end{cases}$$

It is clear that this is a linear order on the set of colors.

On the set of variables $\{x_\gamma(-n) \,|\, \gamma \in \Gamma, \, n \in \mathbb{Z}\} \subset \tilde{\mathfrak{g}}_1$ we define a linear order $<$ by comparing first the degrees, and then the colors of variables:

$$x_\alpha(-i) < x_\beta(-j) \quad \text{if} \quad \begin{cases} -i < -j, \\ i = j \quad \text{and} \quad \alpha < \beta. \end{cases}$$

Since the algebra $\tilde{\mathfrak{g}}_1$ is commutative, we can assume that the variables in monomials from $S(\tilde{\mathfrak{g}}_1)$ are sorted ascendingly from left to right. The order $<$ on the set of monomials is defined as a lexicographic order, where we compare variables from right to left (from the greatest to the lowest one). If $x(\pi)$ and $x(\pi')$ are two monomials,

$$\begin{aligned} x(\pi) &= x_{\gamma_r}(-n_r)x_{\gamma_{r-1}}(-n_{r-1})\cdots x_{\gamma_2}(-n_2)x_{\gamma_1}(-n_1), \\ x(\pi') &= x_{\gamma'_s}(-n'_s)x_{\gamma'_{s-1}}(-n'_{s-1})\cdots x_{\gamma'_2}(-n'_2)x_{\gamma'_1}(-n'_1), \end{aligned}$$

then $x(\pi) < x(\pi')$ if there exists $i_0 \in \mathbb{N}$ such that $x_{\gamma_i}(-n_i) = x_{\gamma'_i}(-n'_i)$ holds for all $i < i_0$, and either $i_0 = r+1 \leq s$ or $x_{\gamma_{i_0}}(-n_{i_0}) < x_{\gamma'_{i_0}}(-n'_{i_0})$.

This monomial order is compatible with multiplication:

Proposition 1. *Let*

$$x(\pi_1) \leq x(\mu_1) \quad \textit{and} \quad x(\pi_2) \leq x(\mu_2).$$

Then

$$x(\pi_1)x(\pi_2) \leq x(\mu_1)x(\mu_2),$$

with the last inequality being strict if any of the first two inequalities is strict.

Proof: By the definition of the order $<$, we compare two monomials so by comparing their greatest elements first. Let $x_{\alpha_1}(-j_1)$, $x_{\alpha_2}(-j_2)$, $x_{\beta_1}(-i_1)$, $x_{\beta_2}(-i_2)$ be the greatest variables in $x(\pi_1)$, $x(\pi_2)$, $x(\mu_1)$, $x(\mu_2)$ respectively. Then $x_{\alpha_1}(-j_1) \leq x_{\beta_1}(-i_1)$ and $x_{\alpha_2}(-j_2) \leq x_{\beta_2}(-i_2)$. The greatest element in $x(\pi)$ is the greater of the two $x_{\alpha_1}(-j_1)$ and $x_{\alpha_2}(-j_2)$; one can assume it to be $x_{\alpha_1}(-j_1)$. Similarly, the greatest element in $x(\mu)$ is the greater of the two $x_{\beta_1}(-i_1)$ and $x_{\beta_2}(-i_2)$. There are two possibilities:

(i) the greatest element of $x(\mu)$ is strictly greater than the greatest element of $x(\pi)$. In that case $x(\pi) < x(\mu)$.

(ii) the greatest element of $x(\mu)$ is equal to the greatest element of $x(\pi)$. Then $x_{\alpha_1}(-j_1) = x_{\beta_1}(-i_1)$, and we can take $x_{\beta_1}(-i_1)$ for the greatest element of $x(\mu)$. We proceed by induction: let $x(\pi'_1)$ and $x(\mu'_1)$ be monomials obtained from $x(\pi_1)$ and $x(\mu_1)$, respectively, by omitting $x_{\alpha_1}(-j_1) = x_{\beta_1}(-i_1)$. Then $x(\pi'_1) \leq x(\mu'_1)$, and we can continue to apply the same procedure to monomials $x(\pi'_1)$, $x(\pi_2)$, $x(\mu'_1)$ and $x(\mu_2)$. After a finite number of steps either the case (i) will occur, or we exhaust monomials $x(\pi_1)$ and $x(\pi_2)$. Both these cases imply $x(\pi) \leq x(\mu)$, and the equality occurs only if both initial inequalities were in fact equalities.

∎

A *degree* of a monomial is the sum of degrees of its variables,

$$\deg(x_{\gamma_r}(-n_r)x_{\gamma_{r-1}}(-n_{r-1})\cdots x_{\gamma_2}(-n_2)x_{\gamma_1}(-n_1)) = -n_1 - n_2 - \cdots - n_r.$$

A *shape* of a monomial is obtained from its colored partition by forgetting colors and considering only the degrees of factors. More precisely, for a monomial $x(\pi)$, the corresponding shape is

$$s_\pi : \mathbb{Z} \to \mathbb{Z}_+, \qquad s_\pi(j) = \sum_{\gamma \in \Gamma} \pi(x_\gamma(-j)).$$

A linear order can also be defined on the set of shapes; we say that $s_\pi < s_{\pi'}$ if there exists $j_0 \in \mathbb{Z}$ such that $s_\pi(j) = s_{\pi'}(j)$ for $j < j_0$.

In the end, for the sake of simplicity, we introduce the following notation:

$$x_{rs}(-j) = x_{\gamma_{rs}}(-j),$$

for $\gamma_{rs} \in \Gamma$.

5. Vertex operator construction of level 1 modules

We use the vertex operator algebra construction of the basic $\tilde{\mathfrak{g}}$-modules (i.e. the standard $\tilde{\mathfrak{g}}$-modules of level 1) (cf. Frenkel and Kac (1980), Segal (1981)). In this section we present only a sketch of the construction, with details to be found in Frenkel, Lepowsky and Meurman (1988), Dong and Lepowsky (1993) or Lepowsky and Lie (2004).

Let $\hat{P}$ be a central extension of P by the finite cyclic group $\langle e^{\pi i/(\ell+1)^2}\rangle$ of order $2(\ell+1)^2$,

$$1 \longrightarrow \langle e^{\pi i/(\ell+1)^2}\rangle \longrightarrow \hat{P} \longrightarrow P \longrightarrow 1.$$

By restriction, one gets a central extension $\hat{Q}$ of Q. Central extension can be chosen such that the corresponding 2-cocycle

$$\epsilon : P \times P \to \langle e^{\pi i/(\ell+1)^2}\rangle$$

satisfies

$$\epsilon(\alpha,\beta)/\epsilon(\beta,\alpha) = (-1)^{\langle\alpha,\beta\rangle} \quad \text{for } \alpha,\beta \in Q.$$

Let

$$c(\lambda,\mu) = \epsilon(\lambda,\mu)/\epsilon(\mu,\lambda) \quad \text{for } \lambda,\mu \in P$$

be the corresponding bimultiplicative, alternating commutator map (cf. Frenkel, Lepowsky and Meurman (1988)).

Inside $\tilde{\mathfrak{g}}$ there is a Heisenberg subalgebra

$$\hat{\mathfrak{h}}_{\mathbb{Z}} = \sum_{n\in\mathbb{Z}\setminus\{0\}} \mathfrak{h} \otimes t^n \oplus \mathbb{C}c.$$

We also introduce subalgebras

$$\begin{aligned} \hat{\mathfrak{h}} &= \mathfrak{h} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c, \\ \hat{\mathfrak{h}}_{\pm} &= \mathfrak{h} \otimes t^{\pm 1}\mathbb{C}[t^{\pm 1}], \end{aligned}$$

and by $\mathbb{C}[P]$ and $\mathbb{C}[Q]$ we denote group algebras of weight and root lattices, respectively. Bases of $\mathbb{C}[P]$ and $\mathbb{C}[Q]$ are $\{e^\lambda \mid \lambda \in P\}$ and $\{e^\alpha \mid \alpha \in Q\}$, respectively.

Consider the induced $\hat{\mathfrak{h}}_{\mathbb{Z}}$-module

$$M(1) = U(\hat{\mathfrak{h}}_{\mathbb{Z}}) \otimes_{\hat{\mathfrak{h}}_+ \oplus \mathbb{C}c} \mathbb{C},$$

where $\mathfrak{h} \otimes \mathbb{C}[t]$ acts trivially on $\mathbb{C}$, and c acts as 1. The module $M(1)$ is irreducible; as a vector space, $M(1)$ is naturally isomorphic to the symmetric algebra $S(\hat{\mathfrak{h}}_-)$ (cf. Frenkel, Lepowsky and Meurman (1988)).

Consider the following tensor products

$$\begin{aligned} V_P &= M(1) \otimes \mathbb{C}[P], \\ V_Q &= M(1) \otimes \mathbb{C}[Q]; \end{aligned}$$

there is a natural inclusion $V_Q \subset V_P$. For simplicity, we often write e^λ instead of $1 \otimes e^\lambda$, and 1 instead of $1 \otimes 1$.

Space V_P carries a $\hat{\mathfrak{h}}$-module structure: $\hat{\mathfrak{h}}_{\mathbb{Z}}$ acts as $\hat{\mathfrak{h}}_{\mathbb{Z}} \otimes 1$ and $\mathfrak{h} \otimes t^0$ acts as $1 \otimes \mathfrak{h}$. The operators $h(0), h \in \mathfrak{h}$ on $\mathbb{C}[P]$ are defined by

$$h(0) \cdot e^\lambda = \langle h,\lambda\rangle e^\lambda$$

for $\lambda \in P$. On V_P we also have the action of $\mathbb{C}[P]$:

$$e^\lambda = 1 \otimes e^\lambda, \quad \lambda \in P,$$

where the latter operator e^λ is a multiplication in $\mathbb{C}[P]$. It will be clear from the context when e^λ denotes a multiplication operator, and when an element of V_P. Define also operators ϵ_λ by

$$\epsilon_\lambda \cdot e^\mu = \epsilon(\lambda,\mu) e^\mu,$$

for $\lambda, \mu \in P$.

For an element $v = h_1(-n_1)\cdots h_r(-n_r)\otimes e^\lambda$ of V_P we define a degree by

$$\deg(v) = -n_1 - n_2 - \cdots - n_r - \frac{1}{2}\langle\lambda,\lambda\rangle.$$

This gives a grading on V_P that is bounded from above.

We use independent commuting formal variables $z, z_0, z_1, z_2, \ldots$. For a vector space V, denote by $V[[z]]$ the space of formal series of nonnegative integral powers of z with coefficients in V. Similarly, denote by $V[[z, z^{-1}]]$ the space of formal Laurent series, and by $V\{z\}$ the space of formal series of rational powers of z with coefficients in V.

Define also one more family of operators $z^h \in (\mathrm{End}\, V_P)\{z\}$ by

$$z^h \cdot e^\lambda = e^\lambda z^{\langle h,\lambda\rangle},$$

for $h \in \mathfrak{h}, \lambda \in P$.

Space V_Q has a structure of vertex operator algebra and V_P is a module for this algebra (cf. Frenkel, Lepowsky and Meurman (1988), Dong and Lepowsky (1993)). Before we define a vertex operator algebra structure on V_Q, define operators

$$\begin{aligned} h(z) &= \sum_{j\in\mathbb{Z}} h(j) z^{-j-1}, \\ E^\pm(h,z) &= \exp\left(\sum_{m\geq 1} h(\pm m)\frac{z^{\mp m}}{\pm m}\right), \end{aligned}$$

for $h \in \mathfrak{h}$. We define vertex operators for all the elements of V_P, rather than just for the elements of V_Q. For the lattice elements, i.e. for the elements $1\otimes e^\lambda = e^\lambda$, set:

$$Y(e^\lambda, z) = E^-(-\lambda, z)E^+(-\lambda, z)\otimes e^\lambda z^\lambda \epsilon_\lambda. \tag{6}$$

Generally, for a homogenous vector $v \in V_P$

$$v = h_1(-n_1)\cdots h_r(-n_r)\otimes e^\lambda,$$

$n_1, \ldots, n_r \geq 1$, set

$$Y(v,z) = {\circ\atop\circ}\left(\frac{\partial_z^{n_1-1}}{(n_1-1)!}h_1(z)\right)\cdots\left(\frac{\partial_z^{n_r-1}}{(n_r-1)!}h_r(z)\right)Y(e^\lambda, z){\circ\atop\circ},$$

where ${\circ\atop\circ}\cdot{\circ\atop\circ}$ is a normal ordering procedure (cf. Frenkel, Lepowsky and Meurman (1988)), meaning that coefficients in the enclosed expression should be rearranged in a way that in each product all the operators $h(m), h \in \mathfrak{h}, m < 0$ are placed to the left of the operators $h(m), h \in \mathfrak{h}, m \geq 0$. This way we get a well defined linear map

$$\begin{aligned} Y : V_P &\to (\mathrm{End}\, V_P)\{z\}, \\ v &\mapsto Y(v,z). \end{aligned}$$

By using vertex operators, we can define a structure of $\tilde{\mathfrak{g}}$-module on V_P. For $\alpha \in R$ set

$$x_\alpha(z) = \sum_{j\in\mathbb{Z}} x_\alpha(j) z^{-j-1} = Y(e^\alpha, z),$$

for a properly chosen root vectors x_α. Actions of $h(j)$ and c have already been defined, and d acts as a degree operator. Then the cosets V_Q and $V_Q e^{\omega_j}, j = 1, \ldots, \ell$ become standard $\tilde{\mathfrak{g}}$-modules of level 1 with the highest weight vectors $v_0 = 1$ and $v_j = e^{\omega_j}, j = 1, \ldots, \ell$, respectively (cf. Frenkel, Lepowsky and Meurman (1988), Dong and Lepowsky (1993)). Moreover,

$$L(\Lambda_0) \cong V_Q,\ L(\Lambda_j) \cong V_Q e^{\omega_j} \text{ for } j = 1, \ldots, \ell$$

and

$$V_P \cong L(\Lambda_0) \oplus L(\Lambda_1) \oplus \cdots \oplus L(\Lambda_\ell).$$

Vertex operators $Y(v,z)$ and $Y(u,z)$, $u,v \in V_P$, satisfy the (generalized) Jacobi identity (cf. Dong and Lepowsky (1993)). It will be of importance to us a variant of that identity in the case when $u = u^* \otimes e^\lambda$, $v = v^* \otimes e^\mu$, for $\lambda \in Q, \mu \in P,\ u^*, v^* \in M(1)$. Then one has

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2) - (-1)^{\langle\lambda,\mu\rangle}c(\lambda,\mu)z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(u,z_1) = \\ = z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y(Y(u,z_0)v,z_2),$$

where $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$ is a formal delta-function (cf. Frenkel, Lepowsky and Meurman (1988), Lepowsky and Li (2004)), and binomial expressions that appear in expansions of delta-functions are understood to be expanded in nonnegative powers of the second variable.

Next, we introduce intertwining operators $\mathcal{Y}$: for $\mu \in P$, $v = v^* \otimes e^\mu$ define

$$\mathcal{Y}(v,z) = Y(v,z)e^{i\pi\mu}c(\cdot,\mu).$$

In this way we obtain a map

$$\begin{aligned} \mathcal{Y} : V_P &\to (\operatorname{End} V_P)\{z\}, \\ v &\mapsto \mathcal{Y}(v,z). \end{aligned}$$

Then we have the (ordinary) Jacobi identity

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)\mathcal{Y}(v,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)\mathcal{Y}(v,z_2)Y(u,z_1) = \\ = z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\mathcal{Y}(Y(u,z_0)v,z_2).$$

For $\mu \in Q$, operators $\mathcal{Y}(v,z)$ are equal to vertex operators $Y(v,z)$. Restrictions of $\mathcal{Y}(v,z)$ are in fact maps

$$\mathcal{Y}(v,z) : L(\Lambda_i) \to L(\Lambda_j)\{z\}, \tag{7}$$

if $\mu + \omega_i \equiv \omega_j \mod Q$. Therefore restrictions of $\mathcal{Y}$ define intertwining operators between standard modules of level 1 (Dong and Lepowsky (1993)).

Consider now the special case when $v = e^\mu$. We are interested in the operators $\mathcal{Y}(e^\mu, z_2)$ from (7) which commute with the action of $\tilde{\mathfrak{g}}_1$, i.e. for which

$$[Y(e^\gamma,z_1),\mathcal{Y}(e^\mu,z_2)] = 0, \quad \text{for all } \gamma \in \Gamma.$$

By the commutator formula for intertwining operators (Dong and Lepowsky (1993)) this is equivalent to

$$Y(e^\gamma,z_0)e^\mu \in V_P[[z_0]],$$

for all $\gamma \in \Gamma$. From (6) one gets

$$Y(e^\gamma,z_0)e^\nu = Ce^{\gamma+\nu}z_0^{\langle\gamma,\nu\rangle} + \underbrace{\cdots}_{\text{higher power terms}} \in z_0^{\langle\gamma,\nu\rangle}V_P[[z_0]], \tag{8}$$

for some $C \in \mathbb{C}^\times$. Therefore, an operator $\mathcal{Y}(e^\mu,z_2)$ commute with $\tilde{\mathfrak{g}}_1$ if and only if

$$\langle\gamma,\mu\rangle \geq 0, \quad \text{for all } \gamma \in \Gamma.$$

In section 8, we describe all $\mu \in P$ that satisfy this relation.

6. Operator $e(\omega)$

For $\lambda \in P$, e^{λ} denotes the multiplication operator $1 \otimes e^{\lambda}$ in $V_P = M(1) \otimes \mathbb{C}[P]$. Set

$$e(\lambda) = e^{\lambda}\epsilon(\cdot, \lambda), \qquad e(\lambda) : V_P \to V_P.$$

Clearly, $e(\lambda)$ is a linear bijection. Its restrictions on standard modules are bijections from one fundamental module $L(\Lambda_i)$ onto another fundamental module $L(\Lambda_{i'})$. From the definition of vertex operators (6), one gets the following commutation relation

$$Y(e^{\alpha}, z)e(\lambda) = e(\lambda)z^{\langle \lambda, \alpha \rangle}Y(e^{\alpha}, z),$$

for $\alpha \in R$. In terms of the components, we have

$$x_{\alpha}(n)e(\lambda) = e(\lambda)x_{\alpha}(n + \langle \lambda, \alpha \rangle), \quad n \in \mathbb{Z}. \tag{9}$$

For $\lambda = \omega$ and $\gamma \in \Gamma$, the relation (9) becomes

$$x_{\gamma}(n)e(\omega) = e(\omega)x_{\gamma}(n + 1).$$

More generally, for a monomial $x(\pi) \in S(\tilde{\mathfrak{g}}_1)$, denote by $x(\pi^+) \in S(\tilde{\mathfrak{g}}_1)$ the monomial corresponding to the partition π^+ defined by $\pi^+(x_{\gamma}(n+1)) = \pi(x_{\gamma}(n))$. We can say that $x(\pi^+)$ is obtained from $x(\pi)$ by raising the degrees of all of its factors by 1. Then

$$x(\pi)e(\omega) = e(\omega)x(\pi^+).$$

7. Difference and initial conditions

Initial conditions for the level 1 standard module $L(\Lambda_i)$ are consequence of a simple observation that monomials from the monomial basis cannot contain elements of degree -1 that act as zero on the highest weight vector v_i of $L(\Lambda_i)$. So, we have to establish for which $\gamma \in \Gamma$, elements $x_{\gamma}(-1)$ annihilate v_i. Then we can exclude from the spanning set (5) all monomials $x(\pi)$ that contain such factors.

Since $v_i = e^{\omega_i}$, for $i = 1, \ldots, \ell$, and $v_0 = 1 = e^0$, relation (8) gives

$$x_{\gamma}(z)v_i = \left(\sum_{j \in \mathbb{Z}} x_{\gamma}(-j)z^{j-1}\right) v_i \in z^{\langle \gamma, \omega_i \rangle}(v_i V_Q)[[z]], \tag{10}$$

and

$$x_{\gamma}(z)v_i = \left(\sum_{j \in \mathbb{Z}} x_{\gamma}(-j)z^{j-1}\right) v_i \in z^{\langle \gamma, \omega_i \rangle}(v_i V_Q)[[z]], \tag{11}$$

for $i = 1, \ldots, \ell$. Since by (4), $\langle \gamma_{rs}, \omega_i \rangle = 1$ if $r \leq i \leq s$, and zero otherwise, by comparing constant terms in (10) and (11), we get

$$x_{\gamma_{rs}}(-1)v_i = \begin{cases} 0, & r \leq i \leq s, \\ Ce^{\gamma_{rs}},\ C \in \mathbb{C}^{\times}, & i = 0, \\ Ce^{\gamma_{rs}+\omega_i},\ C \in \mathbb{C}^{\times}, & \text{otherwise.} \end{cases} \tag{12}$$

For a monomial $x(\pi) \in S(\tilde{\mathfrak{g}}_1^-)$ we say that it satisfies *initial conditions* for $L(\Lambda_i)$ if it does not contain factors of degree -1 that annihilate v_i. We often abbreviate this by saying that $x(\pi)$ satisfies IC for $L(\Lambda_i)$. From (12) we see that $x(\pi)$ satisfies initial conditions for $L(\Lambda_i)$ if the colors of elements of degree -1 lie below the i-th row (in the case $i \leq m$), or, to the left of the i-th column (for $i \geq m$).

Difference conditions will be consequences of relations between operators $x_{\gamma}(z)$. To obtain these, consider the basic module $L(\Lambda_0)$ with highest weight vector $v_0 = 1 = e^0$ (cf. section 5). This is a vertex operator algebra, with 1 as the vacuum element, and $L(\Lambda_i)$ is a module for this algebra. We are looking for relations between vectors of the type

$$x_\gamma(-1)x_{\gamma'}(-1)1, \quad \gamma, \gamma' \in \Gamma.$$

These will in turn induce relations between the corresponding vertex operators on $L(\Lambda_i)$.

From (10) we have

$$x_\gamma(-1)x_{\gamma'}(-1)1 = x_\gamma(-1)e^{\gamma'}.$$

Since

$$\langle \gamma, \gamma' \rangle = \begin{cases} 2, & \gamma = \gamma', \\ 1, & \gamma \text{ and } \gamma' \text{ lie in the same row or column}, \\ 0, & \text{otherwise}, \end{cases}$$

relation (8) implies

$$x_\gamma(-1)x_{\gamma'}(-1)1 = \begin{cases} 0, & \gamma \text{ and } \gamma' \text{ lie in the same row or column}, \\ Ce^{\gamma+\gamma'},\ C \in \mathbb{C}^\times, & \text{otherwise}. \end{cases}$$

Fix two rows $r_1 < r_2$ and two columns $s_2 < s_1$. Note that

$$\gamma_{r_1 s_1} + \gamma_{r_2 s_2} = \gamma_{r_1 s_2} + \gamma_{r_2 s_1}.$$

This gives us

$$x_{r_2 s_2}(-1)x_{r_1 s_1}(-1)1 = C \cdot x_{r_2 s_1}(-1)x_{r_1 s_2}(-1)1,$$

for some $C \in \mathbb{C}^\times$.

We have obtained two types of relations:

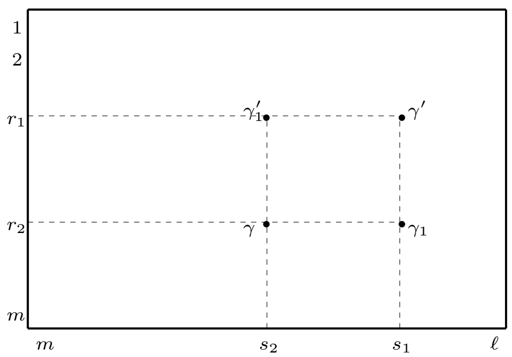
$$x_\gamma(-1)x_{\gamma'}(-1)1 = 0,$$

if γ and γ' lie in the same row/column, and

$$x_\gamma(-1)x_{\gamma'}(-1)1 = C \cdot x_{\gamma_1}(-1)x_{\gamma_1'}(-1)1,$$

if γ, γ_1, γ' and γ_1' are vertices of a rectangle in Γ, as in figure 5.

FIGURE 5. Colors of operators in relations for $L(\Lambda_i)$



Since the algebra $\tilde{\mathfrak{g}}_1$ is commutative, vertex operators $Y(x_\gamma(-1)x_{\gamma'}(-1)1, z)$ are equal to ordinary products of $x_\gamma(z)$ and $x_{\gamma'}(z)$ as Laurent series (cf. Dong and Lepowsky (1993), Lepowsky and Li (2004)). This way we get relations between vertex operators on level 1 modules:

$$x_\gamma(z)x_{\gamma'}(z) = 0, \tag{13}$$

$$x_\gamma(z)x_{\gamma'}(z) = C \cdot x_{\gamma_1}(z)x_{\gamma_1'}(z). \tag{14}$$

Fix $n \in \mathbb{N}$ and consider the coefficients of z^{n-2} in (13) and (14). From the first relation we have

$$0 = \sum_{i+j=n} x_{\gamma}(-i)x_{\gamma'}(-j).$$

In each such sum we identify the minimal monomial with regard to the ordering $<$, which is then called the *leading term* of the relation. This monomial can be expressed in terms of other monomials in the sum, so we can exclude from the spanning set (5) all monomials that contain the leading terms (cf. Primc (1994,2000)). Since all the monomials appearing in the sum are of the same length and of the same degree, the minimal among them has to be of the *minimal shape*, i.e. its factors have to be either of the same degree (for n even), or degrees have to differ only by 1 (for n odd). In the case of n even, there is only one monomial of the minimal shape,

$$x_{\gamma}(-j)x_{\gamma'}(-j), \tag{15}$$

and that is the leading term of the sum above. For n odd, there are two monomials of the minimal shape,

$$x_{\gamma'}(-j-1)x_{\gamma}(-j),\ x_{\gamma}(-j-1)x_{\gamma'}(-j).$$

Following the definition of the order $<$, we next compare the colors of elements. First compare the colors of elements of degree $-j$, and then of elements of degree $-j-1$. Assuming $\gamma < \gamma'$, then the leading term will be

$$x_{\gamma'}(-j-1)x_{\gamma}(-j). \tag{16}$$

Analogously, consider relation (14); we get

$$0 = \sum_{i+j=n} x_{\gamma}(-i)x_{\gamma'}(-j) - Cx_{\gamma_1}(-i)x_{\gamma_1'}(-j).$$

Assume $\gamma < \gamma_1 < \gamma_1' < \gamma'$, as in figure 5. For n even, there are two monomials of the minimal shape:

$$x_{\gamma}(-j)x_{\gamma'}(-j),\ x_{\gamma_1}(-j)x_{\gamma_1'}(-j),$$

and for n odd, there are four of them:

$$x_{\gamma'}(-j-1)x_{\gamma}(-j),\ x_{\gamma}(-j-1)x_{\gamma'}(-j),$$
$$x_{\gamma_1'}(-j-1)x_{\gamma_1}(-j),\ x_{\gamma_1}(-j-1)x_{\gamma_1'}(-j).$$

The leading terms are

$$x_{\gamma_1}(-j)x_{\gamma_1'}(-j) \tag{17}$$

for n even, and

$$x_{\gamma'}(-j-1)x_{\gamma}(-j) \tag{18}$$

for n odd.

We say that a monomial $x(\pi) \in S(\tilde{\mathfrak{g}}_1^-)$ satisfies *difference conditions*, DC for short, if it does not contain any of the leading terms (15)–(18).

Then, by using Proposition 1, we get the following result (cf. Lemma 9.4 in Primc (1994) or Theorem 5.3 in Primc (2000))

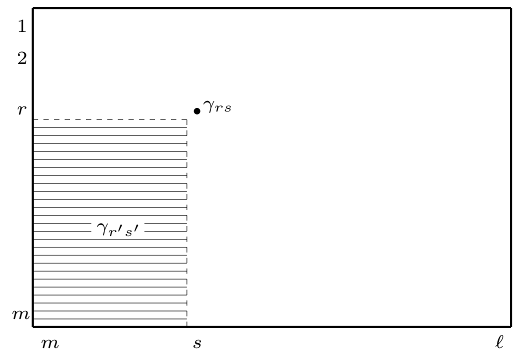
Proposition 2. *The set*

$$\{x(\pi)v_i \,|\, x(\pi) \text{ satisfies IC and DC for } L(\Lambda_i)\} \tag{19}$$

spans $W(\Lambda_i)$.

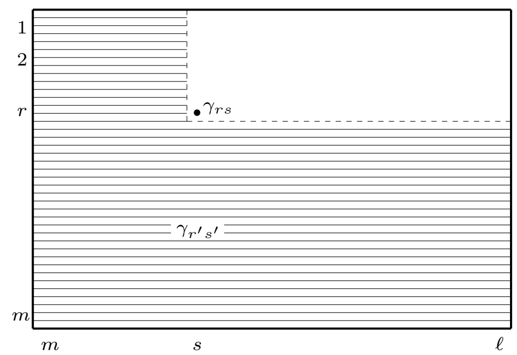
Finally, let's have a closer look at the structure of monomials that satisfy difference and initial conditions for the standard module $L(\Lambda_i)$ of level 1. Assume that a monomial $x(\pi)$ contains elements $x_{rs}(-j)$ and $x_{r's'}(-j)$, and $\gamma_{r's'} \leq \gamma_{rs}$. Then by (15), $\gamma_{r's'}$ and γ_{rs} cannot lie in the same column or row, because otherwise $x(\pi)$ would contain a leading term. Hence $\gamma_{r's'}$ and γ_{rs} are the opposite vertices of a rectangle in Γ. By (17), they have to be upper-right and lower-left vertices of this rectangle, otherwise $x(\pi)$ would again contain a leading term. Since $\gamma_{r's'} \leq \gamma_{rs}$, we conclude that $r' > r$ and $s' < s$, i.e. $\gamma_{r's'}$ must lie in the shaded area of figure 6.

FIGURE 6. Difference conditions: the colors of elements of the same degree



Next, assume that a monomial $x(\pi)$ contains elements $x_{rs}(-j)$ and $x_{r's'}(-j-1)$. Then, by a similar argument as above, one concludes that $r' > r$ or $s' < s$, which is illustrated in figure 7.

FIGURE 7. Difference conditions: the colors of elements of degrees $-j$ and $-j-1$



From these observations we conclude that the colors of elements of the same degree $-j$ inside $x(\pi)$ make a descending sequence as pictured in figure 2; the appropriate row-indices strictly increase, while the column-indices strictly decrease. The colors of elements of degree $-j-1$ also form a decreasing sequence placed below or on the left of the minimal color of elements of degree $-j$.

Initial conditions for $W(\Lambda_i)$ imply that the sequence of colors of elements of degree -1 lies below the i-th row (if $0 \leq i \leq m$), or on the left of the i-th column (for $m \leq i \leq \ell$) (see figure 3).

These considerations also imply the following

Proposition 3. *If $x_\gamma(-j) < x_{\gamma'}(-j') < x_{\gamma''}(-j'')$ are such that monomials $x_\gamma(-j)x_{\gamma'}(-j')$ and $x_{\gamma'}(-j')x_{\gamma''}(-j'')$ satisfy difference conditions, then so does $x_\gamma(-j)x_{\gamma''}(-j'')$, and consequently $x_\gamma(-j)x_{\gamma'}(-j')x_{\gamma''}(-j'')$.*

Hence, under the assumption that factors in monomials are sorted descendingly from right to left, to see if a monomial satisfies difference conditions, it is enough to check difference conditions on all pairs of successive factors in it.

8. Intertwining operators

As we have already seen in Section 5, operators

$$\mathcal{Y}(e^\lambda, z) : L(\Lambda_i) \to L(\Lambda_{i'})\{z\},$$

commute with the action of $\tilde{\mathfrak{g}}_1$ if and only if

$$\langle \lambda, \gamma \rangle \geq 0, \tag{20}$$

for all $\gamma \in \Gamma$.

Define the "minimal" weights that satisfy (20):

$$\begin{array}{ll} \lambda_1 = \omega_1, & \lambda'_m = \omega_m - \omega_{m+1}, \\ \lambda_2 = \omega_2 - \omega_1, & \lambda'_{m+1} = \omega_{m+1} - \omega_{m+2}, \\ \lambda_3 = \omega_3 - \omega_2, & \quad\vdots \\ \quad\vdots & \lambda'_{\ell-1} = \omega_{\ell-1} - \omega_\ell, \\ \lambda_m = \omega_m - \omega_{m-1}, & \lambda'_\ell = \omega_\ell. \end{array} \tag{21}$$

Then relation (4) gives

$$\begin{array}{rcl} \langle \lambda_r, \gamma \rangle & = & \begin{cases} 1, & \text{if } \gamma \text{ lies in the } r\text{-th row,} \\ 0, & \text{otherwise,} \end{cases} \\ \langle \lambda'_s, \gamma \rangle & = & \begin{cases} 1, & \text{if } \gamma \text{ lies in the } s\text{-th column,} \\ 0, & \text{otherwise.} \end{cases} \end{array} \tag{22}$$

It is obvious that every nonnegative $\mathbb{Z}$-linear combination $\lambda \in P$ of these weights also satisfies condition (20) and, consequently, the appropriate intertwining operator $\mathcal{Y}(e^\lambda, z)$ commutes with $\tilde{\mathfrak{g}}_1$. It can easily be shown that a weight $\lambda \in P$ satisfies (20) if and only if λ can be written in this way, for example

$$\begin{array}{rcl} \omega_3 & = & \lambda_1 + \lambda_2 + \lambda_3, \\ \omega_r & = & \lambda_r + \lambda_{r-1} + \cdots + \lambda_1, \quad \text{for } r \leq m, \\ \omega_s & = & \lambda'_s + \lambda'_{j+1} + \cdots + \lambda'_\ell, \quad \text{for } s \geq m, \\ \omega_m & = & \lambda_m + \lambda_{m-1} + \cdots + \lambda_1 = \lambda'_m + \lambda'_{m+1} + \cdots + \lambda'_\ell. \end{array} \tag{23}$$

In the next section we need the following lemma

Lemma 4. *Let $\gamma_{rs} \in \Gamma$. Then*

$$\gamma_{rs} = \lambda_r + \lambda'_s. \tag{24}$$

Proof: By the Cartan matrix of $\mathfrak{g}$, we have

$$\begin{array}{rcl} \alpha_1 & = & 2\omega_1 - \omega_2, \\ \alpha_j & = & -\omega_{j-1} + 2\omega_j - \omega_{j+1}; \quad j = 2, \ldots, \ell - 1 \\ \alpha_\ell & = & -\omega_{\ell-1} + 2\omega_\ell. \end{array}$$

The claim now follows from (4) and (21). ∎

9. Proof of linear independence

Write a monomial $x(\pi) \in S(\tilde{\mathfrak{g}}_1^-)$ as a product $x(\pi) = x(\pi_2)x(\pi_1)$, where $x(\pi_1)$ consists of the elements of degree -1, and $x(\pi_2)$ consists of the elements of lower degree. The main technical tool in the proof of linear independence is the following proposition:

Proposition 5. *Suppose that a monomial $x(\pi)$ satisfies difference and initial conditions for a level 1 standard module $L(\Lambda_i)$. Then there exists a coefficient $w(\mu)$ of an intertwining operator $\mathcal{Y}(e^\mu, z)$*

$$w(\mu) : L(\Lambda_i) \to L(\Lambda_{i'})$$

for some $i' \in \{0, \dots, \ell\}$, such that:

- *$w(\mu)$ commutes with $\tilde{\mathfrak{g}}_1$,*
- *$w(\mu)x(\pi_1)v_i = Ce(\omega)v_{i'}$, for some $C \in \mathbb{C}^\times$,*
- *$x(\pi_2^+)$ satisfies initial and difference conditions for $L(\Lambda_{i'})$,*
- *$x(\pi_1)$ is maximal for $w(\mu)$, i.e. all the monomials $x(\pi')$ that satisfy initial and difference conditions for $L(\Lambda_i)$ and such that $w(\mu)x(\pi')v_i \neq 0$, have their (-1)-part $x(\pi_1')$ smaller or equal to $x(\pi_1)$.*

Proof: Assume $i = 0$; $\Lambda_i = \Lambda_0$, and $v_0 = 1 = e^0$ is the highest weight vector of $L(\Lambda_0)$. Let

$$x(\pi_1) = x_{r_t s_t}(-1) \cdots x_{r_2 s_2}(-1) x_{r_1 s_1}(-1),$$

where $1 \le r_1 < r_2 < \cdots < r_t \le m$, $\ell \ge s_1 > s_2 > \cdots > s_t \ge m$. Then the colors of elements of degree -2 lie either below the r_t-th row, or left of the s_t-th column (see figure 2). Suppose that they lie below the r_t-th row. Since $\langle \gamma_{r_p s_p}, \gamma_{r_q s_q} \rangle = 0$ for $1 \le p < q \le t$, by (8) one has

$$x(\pi_1)v_0 = x_{r_t s_t}(-1) \cdots x_{r_1 s_1}(-1)1 = C_1 \cdot e^{\gamma_{r_1 s_1} + \cdots + \gamma_{r_t s_t}},$$

for some $C_1 \in \mathbb{C}^\times$. By Lemma 4, we have

$$x(\pi_1)v_0 = C_1 \cdot e^{\lambda_{r_1} + \cdots + \lambda_{r_t} + \lambda'_{s_t} + \cdots + \lambda'_{s_1}}.$$

Set

$$\mu = \sum_{\substack{1 \le r < r_t \\ r \notin \{r_1, \dots, r_t\}}} \lambda_r + \sum_{\substack{\ell \ge s > s_t \\ s \notin \{s_1, \dots, s_t\}}} \lambda'_s + \sum_{s=m}^{s_t - 1} \lambda'_s.$$

Weight μ is the sum of all λ_r's, $1 \le r < r_t$, and all λ'_s's, $\ell \ge s \ge m$, such that in the appropriate rows and columns, respectively, there does not lie any color of elements of $x(\pi_1)$. Let $w(\mu)$ be the coefficient of $z^0 = z^{\langle \mu, 0 \rangle}$ in $\mathcal{Y}(e^\mu, z)$. For $\gamma \in \Gamma$, $w(\mu)e^\gamma \neq 0$ if and only if $\langle \mu, \gamma \rangle = 0$, by (8). Because of (22), for a monomial $x(\pi_1')$ consisting of elements of degree -1 and satisfying difference conditions for $L(\Lambda_0)$, vector $x(\pi_1')v_0$ will not be annihilated by $w(\mu)$ if and only if colors of $x(\pi_1')$ lie in the intersection of the rows $\{r_1, \dots, r_t\} \cup \{r_t + 1, \dots, m\}$ and the columns $\{s_1, \dots, s_t\}$. Clearly, $x(\pi_1)$ is the maximal one among such, so if $w(\mu)x(\pi_1')v_0 \neq 0$ then $x(\pi_1') \le x(\pi_1)$.

Note that

$$\mu + \lambda_{r_1} + \cdots + \lambda_{r_t} + \lambda'_{s_t} + \cdots + \lambda'_{s_1} = \sum_{r=1}^{r_t} \lambda_r + \sum_{s=m}^{\ell} \lambda'_s = \omega_{r_t} + \omega.$$

Hence

$$w(\mu)x(\pi_1)v_0 = C_2 e^{\omega_{r_t} + \omega} = Ce(\omega)v_{r_t},$$

for some $C_2, C \in \mathbb{C}^\times$. Since the colors of elements of degree -2 lie below the r_t-th row, the monomial $x(\pi_2^+)$ satisfies difference and initial conditions for $W(\Lambda_{r_t})$.

Hence the operator $w(\mu) : L(\Lambda_0) \to L(\Lambda_{r_t})$ satisfies the statement of the proposition.

If the colors of elements of $x(\pi)$ of degree -2 lie on the left of the s_t-th row instead of below the r_t-th row, then, when constructing μ, one will replace λ'_s's, for $m \leq s < s_t$, with λ_r's, for $s_t < s \leq m$. That way, we get an operator $w(\mu) : L(\Lambda_0) \to L(\Lambda_{s_t})$.

Finally, assume $1 \leq i \leq \ell$; $v_i = e^{\omega_i}$ is the highest weight vector of $L(\Lambda_i)$. The colors of elements of $x(\pi_1)$ lie either below the i-th row, or on the left of i-th column (see figure 3). Then one constructs $\mu \in P$ similarly as before, with an exception that if $i \leq m$, one will not take λ_r's for $r \leq i$, and if $i \geq m$ one will not take λ'_s's for $s \geq i$. For instance, if $i \leq m$ and the colors of elements of degree -2 in $x(\pi)$ are on the left of the s_t-th column, we will set

$$\mu = \sum_{\substack{i<r<r_t \\ r\notin\{r_1,\ldots,r_t\}}} \lambda_r + \sum_{\substack{\ell\leq s>s_t \\ s\notin\{s_1,\ldots,s_t\}}} \lambda'_s + \sum_{r=r_t+1}^{m} \lambda_r.$$

For the operator $w(\mu)$ we take the coefficient of $z^{\langle\mu,\omega_i\rangle}$ in $\mathcal{Y}(e^\mu, z)$. Since $\omega_i = \lambda_1 + \cdots + \lambda_i$, we have

$$\mu + \gamma_{r_1 s_1} + \cdots + \gamma_{r_t s_t} + \omega_i = \omega + \omega_{s_t}.$$

Hence

$$w(\mu)x(\pi_1)v_i = Ce(\omega)v_{s_t},$$

as desired. ■

Proposition 5 enables us to prove linear independence of the set

$$\{x(\pi)v_i \,|\, x(\pi) \text{ satisfies IC and DC for } L(\Lambda_i)\}.$$

We prove this by induction on degree and on order of monomials. The proof is carried out simultaneously for all level 1 standard modules by using the coefficients of intertwining operators.

Assume

$$\sum c_\pi x(\pi)v_i = 0, \tag{25}$$

where all the monomials $x(\pi)$ satisfy difference and initial conditions for $L(\Lambda_i)$ and are of degree greater or equal to some $-n \in \mathbb{Z}$. Fix a monomial $x(\pi)$ in (25) and suppose that

$$c_{\pi'} = 0 \quad \text{for all} \quad x(\pi') < x(\pi).$$

We are going to show that $c_\pi = 0$.

By Proposition 5, there exists an operator $w(\mu)$ such that

- $w(\mu)$ commutes with $\tilde{\mathfrak{g}}_1$,
- $w(\mu)x(\pi_1)v_i = Ce(\omega)v_{i'}, \quad C \in \mathbb{C}^\times$,
- $x(\pi_2^+)$ satisfies IC and DC for $L(\Lambda_{i'})$,
- $w(\mu)x(\pi')v_i = 0 \quad \text{if} \quad x(\pi'_1) > x(\pi_1)$,

where $\Lambda_{i'}$ is another fundamental weight for $\tilde{\mathfrak{g}}$. Applying the operator $w(\mu)$ to (25) gives

$$\begin{aligned} 0 &= w(\mu)\sum c_{\pi'}x(\pi')v_i \\ &= w(\mu)\sum_{\pi'_1>\pi_1} c_{\pi'}x(\pi')v_i + w(\mu)\sum_{\pi'_1<\pi_1} c_{\pi'}x(\pi')v_i + w(\mu)\sum_{\pi'_1=\pi_1} c_{\pi'}x(\pi')v_i. \end{aligned}$$

The first sum becomes 0 after the application of $w(\mu)$, and the second sum also equals to 0 by the induction hypothesis. What is left is

$$0 = w(\mu)\sum_{\pi_1'=\pi_1} c_{\pi'}x(\pi')v_i = \sum_{\pi_1'=\pi_1} c_{\pi'}x(\pi_2')Ce(\omega)v_{i'} = Ce(\omega)\sum_{\pi_1'=\pi_1} c_{\pi'}x(\pi_2'^{+})v_{i'}.$$

Since $e(\omega)$ is an injection, it follows that

$$\sum_{\pi_1'=\pi_1} c_{\pi'}x(\pi_2'^{+})v_{i'} = 0.$$

All monomials $x(\pi_2'^{+})$ satisfy difference conditions because $x(\pi')$ do. If some of them do not satisfy initial conditions for $L(\Lambda_{i'})$, then the corresponding monomial vectors $x(\pi_2'^{+})v_{i'}$ will be equal to 0. Certainly, $x(\pi_2^{+})$ is not among those. We have ended up with a relation of linear dependence on the standard module $L(\Lambda_{i'})$ in which all the monomials are of degree greater or equal to $-n+1$. By the induction hypothesis they are linearly independent, and, in particular, $c_\pi = 0$. We have proved

Theorem 6. *The set*

$$\{x(\pi)v_i \,|\, x(\pi) \text{ satisfies IC and DC for } L(\Lambda_i)\}$$

is a basis of $W(\Lambda_i)$.

10. Bases of standard modules

We follow here the approach of Primc (1994, 2000) to obtain a basis of a standard level 1 module $L(\Lambda_i)$, $i = 0, \ldots, \ell$, for any choice of $\mathbb{Z}$-gradation (1).

Set

$$e = \prod_{\gamma\in\Gamma} e^{\gamma} = e^{\sum_{\gamma\in\Gamma}\gamma}.$$

From Lemma 4 and (23), we have

$$e = e^{m\sum_{j=1}^{m}\lambda_j + (\ell-m+1)\sum_{j=m}^{\ell}\lambda_j'} = e^{(\ell+1)\omega}. \tag{26}$$

The following proposition was proven by Primc (cf. Theorem 8.2. in Primc (1994) or Proposition 5.2. in Primc (2000))

Proposition 7. *Let $L(\Lambda_i)_\mu$ be a weight subspace of $L(\Lambda_i)$. Then there exists an integer n_0 such that for any fixed $n \leq n_0$ the set of vectors*

$$e^n x_{\gamma_1}(j_1)\cdots x_{\gamma_s}(j_s)v_i \in L(\Lambda_i)_\mu,$$

where $s \geq 0$, $\gamma_1, \ldots, \gamma_s \in \Gamma$, $j_1, \ldots, j_s \in \mathbb{Z}$, is a spanning set of $L(\Lambda_i)_\mu$. In particular,

$$L(\Lambda_i) = \langle e\rangle U(\tilde{\mathfrak{g}}_1)v_i.$$

Theorem 8. *Let $L(\Lambda_i)_\mu$ be a weight subspace of a standard level 1 $\tilde{\mathfrak{g}}$-module $L(\Lambda_i)$. Then there exists $n_0 \in \mathbb{Z}$ such that for any fixed $n \leq n_0$ the set of vectors*

$$\{e^n x(\pi)v_i \in L(\Lambda_i)_\mu, \quad x(\pi) \text{ satisfies IC and DC for } W(\Lambda_i)\}$$

is a basis of $L(\Lambda_i)_\mu$. Moreover, for two choices of $n_1, n_2 \leq n_0$, the corresponding bases are connected by a diagonal matrix.

Proof: From Proposition 7 and Theorem 6 it follows that the set above indeed is a basis of $L(\Lambda_i)_\mu$. It is left to prove the second part of the theorem.

In order to see this, we find a monomial $x(\mu) \in U(\tilde{\mathfrak{g}}_1^-)$ and $f \in \mathbb{N}$ such that the following holds:

(i) $e(\omega)^f v_i = Cx(\mu)v_i$, for some $C \in \mathbb{N}$
(ii) f divides $\ell+1$,

(iii) $x(\mu)$ satisfies difference and initial conditions for $W(\Lambda_i)$,

(iv) if a monomial $x(\pi)$ satisfies difference and initial conditions for $W(\Lambda_i)$, then so does a monomial $x(\pi^{-f})x(\mu)$, where π^{-f} is a partition defined by $\pi^{-f}(x_\gamma(-n-f)) = \pi(x_\gamma(-n))$, $\gamma \in \Gamma, n \in \mathbb{Z}$.

Then we have

$$e(\omega)^f x(\pi)v_i = x(\pi^{-f})e(\omega)^f v_i = Cx(\pi^{-f})x(\mu)v_i.$$

Since $e^\omega x(\pi)v_i$ and $e(\omega)x(\pi)v_i$ are proportional, the second part of the theorem follows.

Let $x(\mu) \in U(\tilde{\mathfrak{g}}_1^-)$ be the maximal monomial satisfying difference and initial conditions for $W(\Lambda_i)$ such that its factors are of degree greater or equal to $-f$; we determine the exact value of f later. Let

$$x(\mu) = x_{p_r,q_r}(-n_r)x_{p_{r-1},q_{r-1}}(-n_{r-1})\dots x_{p_2,q_2}(-n_2)x_{p_1,q_1}(-n_1),$$

where factors are decreasing from right to left. Initial conditions imply

$$x_{p_1,q_1}(-n_1) = \begin{cases} x_{1,\ell}(-1), & \text{if } \ i=0, \\ x_{1,\ell}(-2), & \text{if } \ i=m, \\ x_{i+1,\ell}(-1), & \text{if } \ 0<i<m, \\ x_{1,i-1}(-1), & \text{if } \ m<i\le \ell. \end{cases}$$

Difference conditions give

$$x_{p_t,q_t}(-n_t) = \begin{cases} x_{p_{t-1}+1,q_{t-1}-1}(-n_{t-1}), & \text{if } \ 1\le p_{t-1}<m<q_{t-1}\le \ell, \\ x_{1,q_{t-1}-1}(-n_{t-1}-1), & \text{if } \ p_{t-1}=m<q_{t-1}\le \ell, \\ x_{p_{t-1}+1,\ell}(-n_{t-1}-1), & \text{if } \ 1\le p_{t-1}<m=q_{t-1}, \\ x_{1,\ell}(-n_{t-1}-2), & \text{if } \ p_{t-1}=m=q_{t-1}. \end{cases}$$

for $1<t\le r$.

Degrees of elements of $x(\mu)$ are $-1,-2,\dots,-f$, respectively from right to left. Of course, some of the successive elements may have the same degree, and elements of a certain degree may not occur; according to the initial and difference conditions.

From the above observation we also see that the row-indices of colors of elements are going cyclicly over the set

$$(1,2,\dots,m),$$

and the column-indices are going cyclicly over the set

$$(\ell,\ell-1,\dots,m).$$

We choose f so that we stop when we make a "full circle" over both sets of indices. More precisely, we choose f so that the last element $x_{p_r,q_r}(-n_r)$ of $x(\mu)$ is

$$\begin{array}{lll} x_{m,m}(-f+1), & \text{if} & i=0, \\ x_{m,m}(-f), & \text{if} & i=m, \\ x_{m,i}(-f), & \text{if} & i>m, \\ x_{i,m}(-f), & \text{if} & 0<i<m. \end{array} \tag{27}$$

Then r is equal to the smallest common multiple of m and $\ell-m+1$. From (27) and Proposition 3, it is clear that a monomial $x(\pi)$ satisfies difference and initial conditions for $W(\Lambda_i)$ if and only if $x(\pi^{-f})x(\mu)$ satisfies them.

Denote by $x(\mu_j)$ the $(-j)$-part of $x(\mu)$ if there are elements of degree $-j$ in $x(\mu)$, and put $x(\mu_j)=1$ otherwise. Suppose that $x(\mu_j)\neq 1$. Let γ_j be the color of the smallest element of $x(\mu_j)$. Then at least one of the indices of γ_j is equal to m. Denote by i_j the other index of γ_j (of course, $i_j=m$ if $\gamma_j=\gamma_{m,m}$). In case $x(\mu_j)=1$ set $i_j=0$. From (27) it is obvious that $i_f=i$. The same calculation as in the proof of Proposition 5 shows that

$$x(\mu_1)v_i = C_1 e(\omega)v_{i_1},$$

and

$$x(\mu_j^{+j-1})v_{i_{j-1}} = C_j e(\omega) v_{i_j},$$

for some $C_1, \ldots, C_f \in \mathbb{C}^\times$. Hence

$$\begin{aligned} x(\mu)v_i &= x(\mu_f)\cdots x(\mu_1)v_i \\ &= x(\mu_f)\cdots x(\mu_2)C_1 e(\omega)v_{i_1} \\ &= C_1' e(\omega) x(\mu_f^+)\cdots x(\mu_2^+)v_{i_1} = \cdots = \\ &= C e(\omega)^f v_i, \end{aligned}$$

for some $C, C_1, C_1' \in \mathbb{C}^\times$.

It remains to determine f. If $x(\mu_j) \neq 1$ then $x(\mu_j)$ contains exactly one element with one of its indices equal to m. If $x(\mu_j) = 1$ then $x(\mu_{j-1})$ contains $x_{m,m}(-j+1)$. Hence number f counts how many times we have crossed over m while cyclicly moving over the sets of indices $(1, 2, \ldots, m)$ and $(\ell, \ell-1, \ldots, m)$, i.e. f is equal to the total number of cycles we have made (over both sets of indices). Therefore

$$f = \frac{r}{m} + \frac{r}{\ell - m + 1} = r\frac{\ell+1}{m(\ell-m+1)} = \frac{\ell+1}{r'},$$

where $r' = \frac{m(\ell-m+1)}{r} \in \mathbb{N}$. In particular, f divides $\ell + 1$. ∎

As an illustration, we can take a closer look at the case $m = 1$ which was studied in Primc (1994). In this case, Γ is a rectangle with one row and ℓ columns, consisting of elements $\gamma_{11}, \ldots, \gamma_{1\ell}$. Fix a fundamental weight Λ_i. A monomial $x(\pi)$ satisfies initial conditions for $L(\Lambda_i)$ if it does not contain elements $x_{1i}(-1), \ldots, x_{1\ell}(-1)$. If we assume that elements of $x(\pi)$ are decreasing from right to left, then we can say that $x(\pi)$ satisfies difference conditions on $W(\Lambda_i)$ if for any two successive factors $x_{1s}(-j)x_{1s'}(-j')$ of $x(\pi)$ we either have $j \geq j' + 2$, or $j = j' + 1$ and $s < s'$. If we wrote these conditions in terms of exponentials $\pi(x_\gamma(-j)), \gamma \in \Gamma, j \in \mathbb{N}$, we would obtain a special case of $(k, \ell+1)$-admissible configurations, for $k = 1$ (cf. Feigin, Jimbo, Loktev, Miwa and Mukhin (2003), Trupčević (2009)).

We construct a periodic tail $x(\mu)$ as in the proof of Theorem 8. We obtain

$$x(\mu) = \begin{cases} x_{1,1}(-\ell)\cdots x_{1,\ell-1}(-2)x_{1,\ell}(-1), & \text{if} \quad i = 0, \\ x_{1,1}(-\ell-1)\cdots x_{1,\ell-1}(-3)x_{1,\ell}(-2), & \text{if} \quad i = 1, \\ x_{1,i}(-\ell-1)\cdots x_{1,\ell}(-i-1)x_{1,1}(-i+1)\cdots x_{1,i-1}(-1), & \text{if} \quad 2 \leq i \leq \ell. \end{cases}$$

which is the maximal monomial that satisfies initial and difference conditions for $L(\Lambda_i)$ and has elements of degree greater or equal to $-\ell - 1$. Since

$$\begin{aligned} x_{1,\ell}(-1)v_0 &= C_0 e(\omega) v_\ell, \\ x_{1,j-1}(-1)v_j &= C_j e(\omega) v_{j-1}, \quad j = 2, \ldots, \ell, \\ v_1 &= C_1 e(\omega) v_0, \end{aligned}$$

for some $C_0, \ldots, C_\ell \in \mathbb{C}^\times$, we see that

$$x(\mu)v_i = C e(\omega)^{\ell+1} v_i,$$

for some $C \in \mathbb{C}^\times$.

Also, it is clear that a monomial $x(\pi)$ satisfies initial and difference conditions for $L(\Lambda_i)$ if and only if $x(\pi^{-\ell-1})x(\mu)$ satisfies initial and difference conditions for $L(\Lambda_i)$.

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Department of Mathematics, University of Zagreb, Bijenička 30, Zagreb, Croatia
E-mail address: gtrup@math.hr