NEW METHOD FOR REDUCING SHARP CORNERS IN CARTOGRAPHIC LINES WITH AREA PRESERVATION PROPERTY

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ABSTRACT: In this paper an area preservation function for modification of polylines is presented. For given three consecutive points T_i, T_{i+1} and T_{i+2} in polyline the function returns four consecutive points T_i, *Q*, *S* and T_{i+2}. There are four unknowns: x_Q, y_Q, x_S and y_S, therefore four independent constraints are necessary. The first is area preservation, i.e., the area of the triangle $\Delta T_i T_{i+1} T_{i+2}$ is equal to the area of the quadrilateral $T_i QST_{i+2}$. Other three constraints are chosen to ensure simplicity and applicability. To ensure that new segments are not too long or too short the lengths of new segments are chosen to be equal, i.e. $\overline{T_i Q} = \overline{QS} = \overline{ST_{i+2}}$. We use fourth constraint to define the angles among new segments. We define that smaller angle of two in points *Q* and *S* gets its maximum possible value. This will be true when the angles in points *Q* and *S* are the same. In this way, $T_i QST_{i+2}$ form an isosceles trapezoid. To find its elements and the coordinates of the points *Q* and *S* the fourth order polynomial has to be solved. We prove that there is always one and only one solution to the problem. The solution is given in closed form using Ferrari's method. Using that function, we should be able to reduce sharp corners in polylines which result from the generalization process. The result of such an application is presented.

Keywords: area preservation, polyline, cartographic generalization, smoothing sharp corners.

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1. MOTIVATION

In [8] the algorithm for line generalization with area preservation property is presented. The application of the algorithm in cartography gave good results. The idea of the algorithm is to replace three consecutive segments in polyline with two consecutive segments in polyline (first and last point of the segments remain the same) in a way to preserve the areas bounded by polylines. After lines are generalized by that or some other algorithms, the sharp corners appear in polyline even when they are not part of the original polyline. This sharp corners degrade the visual properties of the lines which are important in cartography. The example of such phenomenon is on Fig. 1. On the left is the original polyline, and on the right is the polyline resulted from generalization by the area preserving algorithm [8].

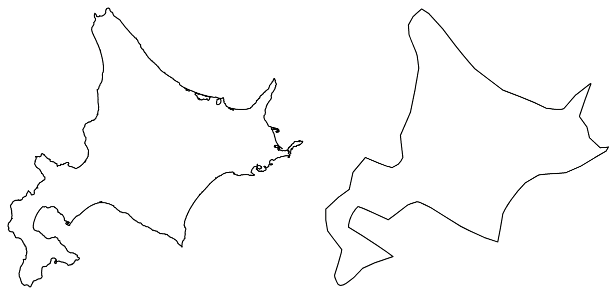

Fig. 1. The example of sharp corners which appear in generalized polylines. The original polyline (left) and generalized polyline (right)

In [8] the inverse area-preserving function which will give three consecutive segments for two consecutive segments in polyline is proposed. Such approach could be used for smoothing the sharp corners. We consider that the area preservation property of the line generalization in cartography is of importance, and for this inverse function the same property should be given. This way the complete process of line generalization by this two functions would preserve areas.

2. AREA IN CARTOGRAPHIC LINE GENERALIZATION

Automatized line generalization is a topic of great interest during last 50 years. According to [10] one of the first algorithms is that of Ivanov from 1965 [2]. Since then numerous algorithms are defined and used. In [6] or [10] an overview of some more popular methods can be found. The whole process of cartographic generalization is difficult to define in an exact way. The human role is of great importance for the final estimation of the success of an automated process [4]. McMaster [5] gave an overview of different statistical measures which can be used to evaluate the results of line generalization.

The shape of a line can not be exactly defined or preserved (in that case, there would be no generalization). Most of the methods are based on the best possible preservation of the "line shape" analysing and preserving important or critical points [4]. A discussion of this idea can be also found in [7].

The property which can be preserved is the ratio of areas before and after line generalization. Williams [11] considers modifications of polylines which lead to area preservation after the simplification or enhancement of polylines. The two proposed algorithms move points in a manner similar to offseting. This results with areas same as original or of some other given values. Bose et. al. [1] give approximations of polylines with three area constraints. Since the approximations are limited to subset of original vertices, they investigate the area error and find optimal approximations. They also consider the problem of existence of approximation for which the area error equals zero. There are also other approaches which takes area into account during generalization, e.g. that of Visvalingam and Whyatt [9].

3. DEFINITIONS

Let Π be a plane with a rectangular Cartesian coordinate system. The ordered pair of coordinates (x, y) define the point in the plane, and it is going to be designated as $T(x, y)$.

Let $V = \{T_i(x_i, y_i) \in \Pi; i = 1,2,...,n\}$ be the ordered set of n points in the plane Π such that $T_j \neq T_{j+1}$, $j = 1,2,...,n-1$, $T_j \neq T_{j+2}$, $j = 1,2,...,n-2$ and $n \geq 3$. The set V is called the set of vertices.

Let us define the set

$$S_j = \begin{cases} T(x, y) \in \Pi; \\ (y - y_j)(x_{j+1} - x_j) = (y_{j+1} - y_j)(x - x_j) \\ \min(x_j, x_{j+1}) \leq x \leq \max(x_j, x_{j+1}) \\ \min(y_j, y_{j+1}) \leq y \leq \max(y_j, y_{j+1}) \end{cases}.$$

The set S_j is called the segment. The segment represents the line in the plane with endpoints T_j and T_{j+1}.

Let $P = \{S_j; j = 1,2,...,n-1\}$ be the ordered set of segments. The set P represents the polyline.

If $T_1 \neq T_n$, the polyline is open, otherwise, $T_1 = T_n$, and the polyline is closed (Fig. 2). It should be noted that according to the definition of the set V, the polyline can not have two identical consecutive vertices nor identical even or odd consecutive vertices and it must have at least three vertices. According to our definition, the polyline can intersect itself (Fig. 3).

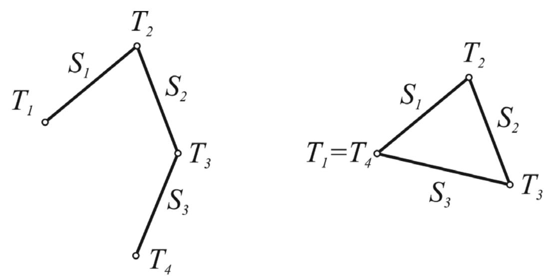

Fig. 2. Open and closed polyline

A polyline is one of basic geometric elements used in digital cartography, GIS and spatial databases. It is used for approximation (more or less simplified representation) of different objects in reality (borders, rivers, contours, transportation, etc.) or phenomena (isolines, planned routes, graticule, etc.).

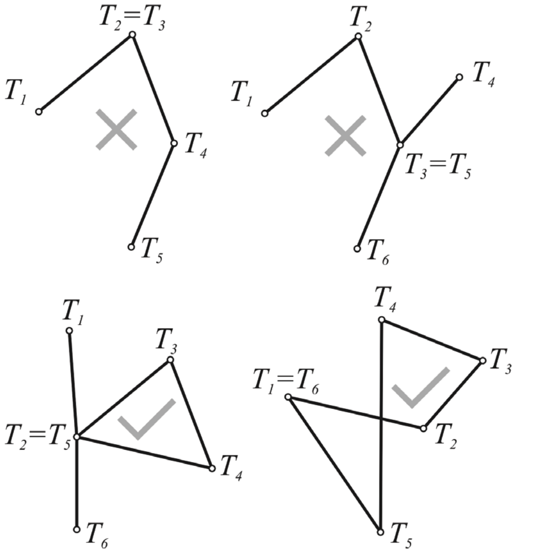

Fig. 3. Non-allowed and allowed cases of polylines according to our definition

When working with polylines, there is a need to approximate polylines with others. Tutić and Lapaine [8] gave the method for line generalization with area preservation property. In this paper the original polyline is replaced with more complex polyline according to number of vertices. but the area is preserved and the new polyline has smoothed sharp corners to the extent defined by user parameters.

4. SMOOTHING FUNCTION

In this section the smoothing function will be defined which will be used for reducing sharp corners in polyline with area preservation property.The sharp corner in polyline is set of two consecutive segments which form the angle less than some value ε (Fig. 4).

It is known that the sum of the angles in simple and closed polygon equals $180n-360$, where n is the number of the sides of polygon and by the simple we mean that no segments in polygon intersects with one another except in vertices. If we add constraint that the smallest angle in simple closed polygon has the maximum value then we have the polygon with all interior angles equal to $180-360/n$. If we want to replace two segments forming the angle α with segments which will form angles greater than α, one way to do this is by replacing two segments with three or more segments. It is sometimes possible to increase the angle α only by moving the middle point along the line parallel to line through first and third point. This case is not considered here but could be introduced in future research.

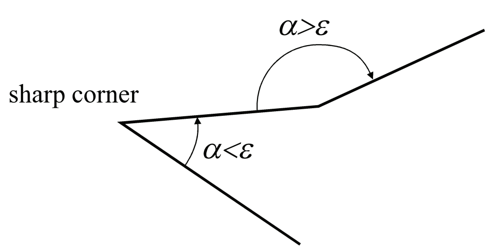

Fig. 4. Sharp corner in polyline

Line from the first to last (third) point of two consecutive segments form the triangle. Let the line $\overline{T_i T_{i+2}}$ be the base a of the triangle. The area of the triangle is P. The angle opposite to the base a is α (Fig. 5).

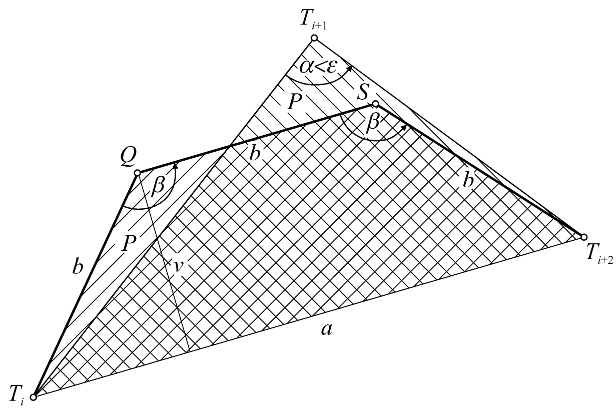

Fig 5. The constraints for smoothing function

The triangle will be replaced by quadrangle and by that the less sharp corners can be formed in the polyline. We want to preserve area. The area of the quadrangle has to be equal to the area of triangle. One side of the quadrangle is given as a, and the additional constraints for the lengths of the other three sides have to be applied. It is important to avoid generation of too long or too short segments. One simple constraint is to have the quadrangle with all three sides (except the base a) of the same length. That way we are sure that the shortest side must be greater than $a/3 > 0$. In the definition of polyline we have that $a > 0$.

To define coordinates of two unknown points we need at least four constraints. One is area preservation, other two are for the lengths of the sides, i.e., $\overline{T_iQ} = \overline{QS}$ and $\overline{QS} = \overline{ST_{i+2}}$. The idea is to form new angles as large as possible. That way we assure that new segments will not form new sharp angles. Let say that the smaller of two new angles is maximal. This will be true when these two new angles are equal. The constraints above define the quadrangle as isosceles trapezoid (Fig. 5).

5. CALCULATION OF THE ELEMENTS OF ISOSCELES TRAPEZOID

As we already defined the first and third point of two consecutive segments are on the distance a and the triangle has the area P. Let other three sides of the quadrangle be b (Fig. 5). Now, the area of the trapezoid is:

$$P = \frac{a+b}{2} v,$$

where v is the height of the trapezoid. First, using the Pitagora's theorem we can write

$$v = \sqrt{b^2 - \left(\frac{a-b}{2}\right)^2}.$$

After modification we get

$$v = \frac{1}{2}\sqrt{(a+b)(3b-a)}.$$

The last expression can be substituted in the equation for the area of trapezoid and after squaring we get

$$16P^2 = (a+b)(3b-a). \qquad (1)$$

We see that it must be $b \geq \frac{a}{3}$. Now we can introduce the substitution $z = 3b - a$, $z \geq 0$. After expressing b using z and substitution in (1) we get

$$432P^2 = (4a+z)^3 z.$$

Now we can form the function $f(z) = (4a+z)^3 z - 432P^2$. First we should note that $f(0) = -432P^2 \leq 0$ and $f'(z) = 4(4a+z)^2(a+z)$. Because $f'(z) > 0$ for $z \geq 0$ we can conclude that function $f(z) = (4a+z)^3 z - 432P^2$ for $z \geq 0$ is increasing so it must have only one root. The roots of the polynomial $f(z) = (4a+z)^3 z - 432P^2$ which is the quartic can be found using the known method of Ferrari. Using Ferrari's method the roots can be expressed as:

$$z = -3a \pm \sqrt{a^2+B} \mp \sqrt{2a^2 - B \pm \frac{2a^3}{\sqrt{a^2+B}}},$$

where

$$B = \frac{1}{2}U - 72\frac{P^2}{U} \quad \text{and}$$

$$U = \left(432P^2\left(\sqrt{a^4+16P^2} - a^2\right)\right)^{\frac{1}{3}}.$$

From the four roots we have to find the one which satisfies $z \geq 0$. We already show the uniqueness of that root. We can do that by inspecting the roots with given values for a and P, e.g. $a = 1$ and $P = 1$. The values of the all for roots for $a = 1$ and $P = 1$ are:

$$z_1 = -3a + \sqrt{a^2+B} + \sqrt{2a^2 - B + \frac{2a^3}{\sqrt{a^2+B}}},$$

$$z_1 = 2,$$

$$z_2 = -3a + \sqrt{a^2+B} - \sqrt{2a^2 - B + \frac{2a^3}{\sqrt{a^2+B}}},$$

$$z_2 = -7.8101,$$

$$z_3 = -3a - \sqrt{a^2+B} + \sqrt{2a^2 - B - \frac{2a^3}{\sqrt{a^2+B}}},$$

$$z_3 = -3.0949 + 4.2518\mathrm{i},$$

$$z_4 = -3a - \sqrt{a^2+B} - \sqrt{2a^2 - B - \frac{2a^3}{\sqrt{a^2+B}}},$$

$$z_4 = -3.0949 - 4.2518\mathrm{i}.$$

The root

$$z = -3a + \sqrt{a^2 + B} + \sqrt{2a^2 - B + \frac{2a^3}{\sqrt{a^2 + B}}}$$

is the one we will use as the solution.
Now when we know z we can easily calculate the length b of the new sides in trapezoid

$b = \frac{z + a}{3}$ and its height $v = \frac{2P}{a + b}$.

6. CALCULATION OF THE COORDINATES OF THE POINTS *Q* AND *S*

When the elements of the trapezoid are known we can find the coordinates of the new points Q and S (Fig. 6).

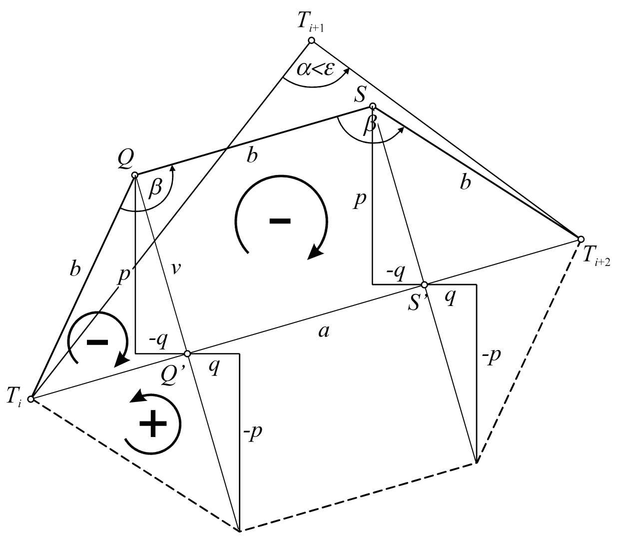

Fig. 6. Determination of the coordinates of points Q and S

The coordinates of the Q' and S' can be easily found using the rule of similar triangles

$$x'_Q = (x_{i+2} - x_i)\left(1 - \frac{b}{a}\right) + x_i ,$$

$$y'_Q = (y_{i+2} - y_i)\left(1 - \frac{b}{a}\right) + y_i ,$$

$$x'_S = (x_{i+2} - x_i)\left(1 + \frac{b}{a}\right) + x_i ,$$

$$y'_S = (y_{i+2} - y_i)\left(1 + \frac{b}{a}\right) + y_i .$$

The values for q and p can be found solving the system:

$$p^2 + q^2 = v^2 , \quad p(y_{i+2} - y_i) + q(x_{i+2} - x_i) = 0$$

This system always has the two solutions which can be interpreted as the intersection of the circle with center in the origin and the straight line through the origin. The solution are the two pairs of values for p and q.

$$q = \pm \frac{v}{a}(y_{i+2} - y_i) , \quad p = \mp \frac{v}{a}(x_{i+2} - x_i) .$$

Between the two we have to choose one which will give the orientation of the trapezoid same as of the triangle. By the orientation we mean clockwise or counter-clockwise order of vertices. To choose the right values for q and p we can use the function [3]:

$$F(T_1, T_2, T_3) = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$ with the property

that $F(T_1, T_2, T_3) > 0$ if the triangle $\Delta T_1 T_2 T_3$ is of counter-clockwise orientation and $F(T_1, T_2, T_3) < 0$ if the triangle is of clockwise orientation.

Let us presume $q = \frac{v}{a}(y_{i+2} - y_i)$ and $p = \frac{v}{a}(x_{i+2} - x_i)$. Then the coordinates of the points Q and S are:

$x_Q = x'_Q + q$, $y_Q = y'_Q + p$ and

$x_S = x'_S + q$, $y_S = y'_S + p$.

If $F(T_i, T_Q, T_{Q'}) \neq F(T_i, T_{i+1}, T_{i+2})$, i.e., the orientation of the triangles $\Delta T_i T_Q T_{Q'}$ and $\Delta T_i T_{i+1} T_{i+2}$ are not equal then

$x_Q = x'_Q - q$, $y_Q = y'_Q - p$ and

$x_S = x'_S - q$, $y_S = y'_S - p$.

Now the smoothing function is completely defined. It takes three consecutive points in polyline and returns four consecutive points giving the less prominent corners and preserving the area.

7. APPLICATION OF THE SMOOTHING FUNCTION

The input parameters of the smoothing function are three consecutive points in polyline and the angle threshold ε which defines whether an angle in vertex is considered sharp or not. If $\alpha > \varepsilon$ the smoothing function returns same points as input, otherwise it returns four points. Recursion can be applied on new points until no sharp corners are found in polyline. For polylines that have more then three points the order of triplets of points sent to the smoothing function has to be defined. It must be ensured that all vertices get tested for sharpness. We can imagine several approaches to this.

In our implementation of the smoothing function we defined the following procedure:

1. First we test if the polyline is closed. If true then to the end of point list the second point is added. That way we ensure that the angle in the first vertex gets tested. If the polyline is not closed go to step 2.
2. Pointer is set to the first point.
3. The pointed point together with next two points are sent to the smoothing function.
4. If we detect sharp angle, i.e., $\alpha < \varepsilon$ compute the function. The point T_{i+1} is replaced with two new points Q and S. If $\alpha \geq \varepsilon$ go to step 7.
5. If the polyline is closed and pointer is set on first point, last point in the list is replaced with Q.
6. If the polyline is closed and pointer is set on point before last point, first point in the list is replaced with point S. Go to step 2.
7. Advance pointer to next point until second point after pointer is not the last point.

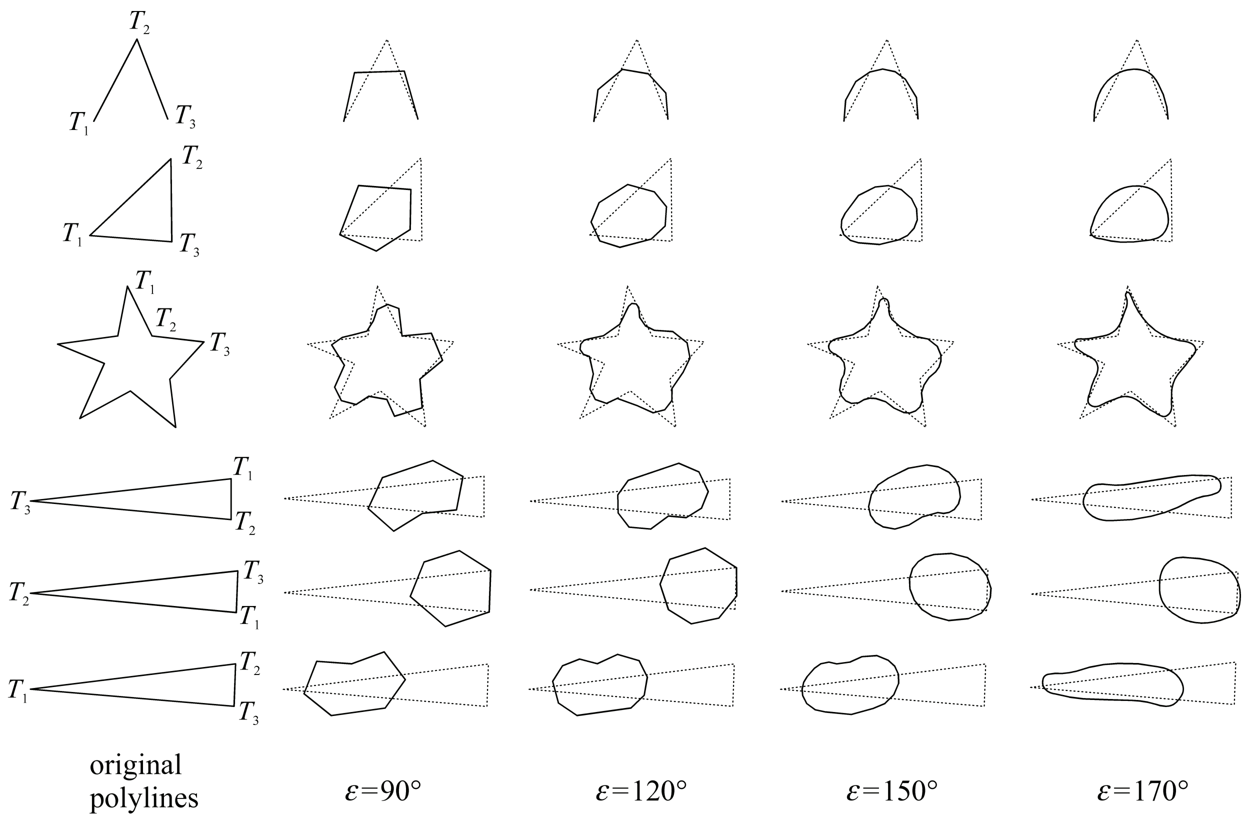

Fig. 7. Examples of application of the smoothing function

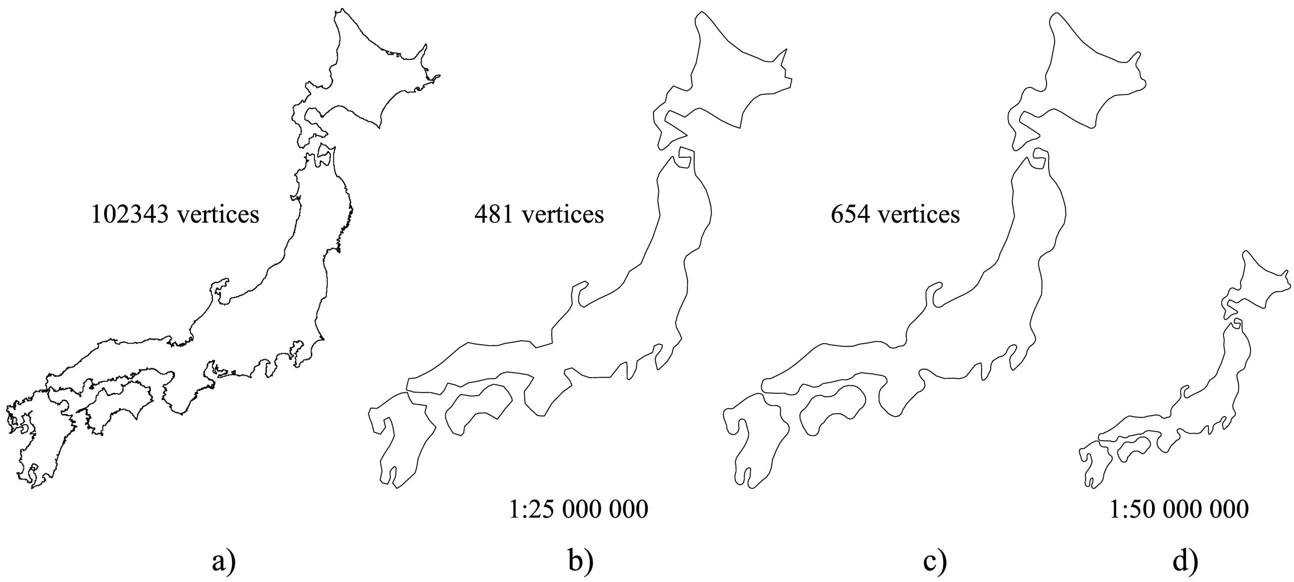

Fig. 8. The application of the smoothing function to the generalized coastline of four largest islands of Japan; (a) source polylines, GSHHS – A Global Self-consistent, Hierarchical, High-resolution Shoreline Database (http://www.ngdc.noaa.gov/mgg/shorelines/gshhs.html), level F; (b) generalized polylines with an area preservation algorithm [8]; (c) smoothed polyline (b) with $\varepsilon = 150°$ and no limit on segment lengths; (d) polyline (c) in more appropriate map scale.

The implementation is done inside GRASS GIS [1]. GRASS GIS has the module *v.generalize* which already implements several algorithms for line simplification and smoothing. The algorithm [8] is also added to this module. GRASS is geoinformation software published under GNU General Public License and the core modules are written in C.

Fig. 7 shows with dashed lines the original polylines, and with continuous lines the smoothed polylines. As can be seen, the application of such smoothing function give results which depend on order of points.

8. APPLICATION OF THE SMOOTHING FUNCTION FOR CARTOGRAPHIC LINE GENERALIZATION

The motivation for this smoothing function was to reduce sharp angles in polylines which result from line generalization. The property of area preservation was applied to use this function together with algorithm for line generalization with the same property of area preservation [8] (Fig. 8). That does not mean that it cannot be applied on polylines generalized by other algorithms.

When applied for cartographic line generalization some additional input parameters are useful. Here we will add a parameter for maximal length of segments. This way the application of the smoothing function is limited only to segments whose length is less then the given threshold. This allows the cartographer to further define the way the polylines are treated. The example when this can be useful is the state border lines which spread along meridians or parallels and form sharp angles. In that case we want to preserve such angles. On contrary, along borders which follow natural objects such as rivers we need smoothed polylines.

9. CONCLUSION

This paper gives the method for smoothing the sharp angles in polylines. Polylines with sharp angles often result from automatic line

[1] GRASS GIS – Geographic Resources Analysis Support System, http://grass.itc.it

generalization in cartography. The authors defined the area preserving algorithm for automatic line generalization with area preservation property (Tutić and Lapaine 2009). Application of that algorithm can result with sharp angles in polylines. Sharp angles in representation of objects which are smooth (coastlines, contours, rivers etc.) have the negative impact on visual quality in cartography. The method for smoothing the sharp angles in this paper has the property of area preservation and is based on adding new segments (points) in polyline. The area preservation property is imposed to keep that property in generalized polylines. The smoothing function takes two consecutive segments and returns three consecutive segments (which form the isosceles trapezoid when closed). The recursive application of such smoothing function may result in polylines which do not have prominent sharp angles.

In cartographic line generalization the application of this smoothing function has the primary purpose of better visual properties of lines, without degrading basic shape. This can be accomplished by appropriate values of the angle and length threshold. The application and given example proves that such application is possible and useful.

It should be mentioned that described method is not the only possible approach to this problem. The proposed method is simple enough to be efficiently applied to large datasets which are common in spatial databases and cartography. It also gives acceptable results.

Both algorithms, the one presented here and the one in [8] are not free from generating self-intersections and intersections of polylines. This will be main concern of future research.

NOTE: Interested in the source code of the modified module *v.generalize* for GRASS GIS should contact the authors.

REFERENCES

[1] Bose, P., Cabello, S., Cheong, O., Gudmundsson, J., van Kreverd, M. and Speckmann, B. *Area-Preserving Approximations of Polygonal Paths.* Journal of Discrete Algorithms, **4** (4), 554-566 (2006).

[2] Ivanov, V.V. *O nekotorih vozmožnostjah avtomatizacii topografičeskih kart.* Geodezija i kartografija, 1, 62-66 (1965).

[3] Lapaine, M. and Frančula, N. *Točka u poligonu.* Geodetski list, 55 (3), 207-222 (2001).

[4] Mackannes, W.A., Ruas, A. and Sarjakovski, L.T. (eds.) *Generalisation of Geographic Information: Cartographic Modelling and Applications*. Elseiver, Amsterdam (2007).

[5] McMaster, R.B. *A Statistical Analysis of Mathematical Measures for Linear Simplification*. The American Cartographer, **13** (2), 103-116 (1986).

[6] McMaster, R.B. *Automated Line Generalization*. Cartographica, **24** (2), 74-11 (1987).

[7] Thapa, K. *A Review of Critical Points Detection and Line Generalization Algorithms.* Surveying and Mapping, **48** (3), 185-205 (1988).

[8] Tutić, D. and Lapaine, M. *Area Preserving Cartographic Line Generalization.* Cartography and Geoinformation, **8** (11), 84-100 (2009).

[9] Visvalingam, M. and Whyatt, J.D. *Line Generalization by Repeated Elimination of Points.* The Cartographical Journal, **30** (1), 46-51, (1993).

[10] Vučetić, N. *Generalizacija linijskih elemenata karte po kriteriju maksimalne sličnosti.* Dissertation. University of Zagreb, (2001).

[11] Williams, R. *Preserving Area of Regions.* Computer Graphics Forum, **6** (1), 43-48 (1987).

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