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SOME ASPECTS OF GEOMETRIC STIFFNESS MODELLING IN THE HYDROELASTIC ANALYSIS OF SHIP STRUCTURES

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Summary

Hydroelastic analysis of ship structures is a complex task of determining the interaction between the structure motion and vibrations on the one hand and water on the other. In the governing equation of motion, the unified restoring and geometric stiffness play an important role. The hydroelastic problem is solved by the mode superposition method based on the finite element technique. Simplified modal geometric stiffness is derived by neglecting the rotational degree of freedom in the shell finite elements. The influence of such a simplification on accuracy is analysed in the case of beam flexural vibrations, where distributed mass is also modelled by a set of lumped masses. Based on good results obtained by both the direct calculation and some commercial software packages, the same approach is used for beam buckling analysis. The performed analysis shows that the use of a simplified geometric stiffness matrix yields quite good results and its application in the hydroelastic analysis of ship structures is acceptable.

Key words: *structural analysis, FEM, beam model, natural vibrations, buckling*

1. Introduction

In hydroelasticity of the ship and offshore structures, as in other branches of engineering science, the finite element method (FEM) represents a widely used numerical tool. A definition of stiffness, mass and damping matrices is required for the constitution of a mathematical model [1]. Determination of the unified restoring and the geometric stiffness is complex due to its nature, but it was noticed that ignoring the rotational degree of freedom (d.o.f.) could significantly simplify the problem [2]. Although there are numerous papers related to the finite element method and its application, to the authors' knowledge, there is no clear answer to the question whether rotations could be neglected in the mass and geometric stiffness matrices or not if the acceptable analysis accuracy level is to be retained.

FEM is a powerful tool in structural analysis and there are many FE software packages currently at our disposal. A typical package for FEM analysis consists of a preprocessor, a solver and a postprocessor. Based on a structural model, global stiffness and mass matrices are generated in order to perform a natural vibration analysis. Commercial software packages offer automatic matrix generation, but it is important to notice that mass can be assumed either as continuous or it can be treated as a set of lumped masses in nodes of finite elements. In the latter case, mass rotation is usually ignored, which could influence the results.

From a mathematical point of view, buckling and natural vibrations are similar eigenvalue problems. However, due to the dependency of geometric stiffness on imposed load, buckling is more complicated. Because of that, a more rational approach would be to investigate how the lumped mass assumption influences the results of natural vibration analysis, and then to use similar discretization of axial load in the analysis of buckling. Furthermore, natural vibration analysis of simple models can be done very quickly using commercial software packages. Within this investigation, a simple beam model is used since it is easy to generate and its analytical solution is available in literature. Only the first few natural modes required for the hydroelastic analysis are considered.

2. Modal geometric stiffness

A hydroelastic analysis of ship structures is performed by the mode superposition method, [3]. The governing dynamic equilibrium equation is comprised of the ship mass and added mass matrices, structural and water damping matrices, a structural stiffness matrix and the unified restoring and the geometric stiffness matrix that have some common terms, [2].

The geometric stiffness is defined by the stresses in calm sea and dry natural modes in the global X, Y, Z coordinate system, and can be written in the index notation [4] as:

$$k_{ij}^{G} = \iiint_{V} \Sigma_{kl} H_{m,k}^{i} H_{m,l}^{j} \mathrm{d}V \,, \tag{1}$$

where Σ_{kl} is the stress tensor, $H_{m,k}^{i}$ and $H_{m,l}^{j}$ are k and l derivatives of the m component of natural modes $\mathbf{H}^{i}$ and $\mathbf{H}^{j}$, respectively, and V is the structure volume.

Formula (1) can also be written in the matrix notation, for easier coding, as:

$$k_{ij}^{G} = \iiint_{V} \Big(\big\langle \Lambda\big(H_x^i\big)\big\rangle [\Sigma] \big\{\Lambda\big(H_x^j\big)\big\} + \big\langle \Lambda\big(H_y^i\big)\big\rangle [\Sigma] \big\{\Lambda\big(H_y^j\big)\big\} + \big\langle \Lambda\big(H_z^i\big)\big\rangle [\Sigma] \big\{\Lambda\big(H_z^j\big)\big\} \Big) \mathrm{d}V \,, \tag{2}$$

where $[\Sigma]$ is the stress matrix and

$$\{\Lambda(.)\} = \begin{Bmatrix} \dfrac{\partial(.)}{\partial X} \\ \dfrac{\partial(.)}{\partial Y} \\ \dfrac{\partial(.)}{\partial Z} \end{Bmatrix}, \qquad \langle \Lambda(.) \rangle = \{\Lambda(.)\}^{T} \tag{3}$$

is the differential operator. The integration of Eq. (2) is performed numerically per volume of discretisized finite elements. For that purpose, it is necessary to transform all the involved quantities from the global to the local x, y, z coordinate system by the matrix of directional coefficients $[c]$, i.e. cosines between the local and global axes, [4]. A characteristic of matrix $[c]$ is

$$[c]^{-1} = [c]^{T} \tag{4}$$

so that

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = \left[c\right]^T \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}. \tag{5}$$

The relation (5) can also be applied for displacements

$$\begin{Bmatrix} H_x \\ H_y \\ H_z \end{Bmatrix} = \left[c\right]^T \begin{Bmatrix} h_x \\ h_y \\ h_z \end{Bmatrix}, \tag{6}$$

where h_x, h_y, h_z are components of a mode in the local coordinate system. One can prove that the following relation is also valid for differential operators in the global and the local system

$$\left\{\Lambda(.)\right\} = \left[c\right]^T \left\{\lambda(.)\right\}, \tag{7}$$

where

$$\left\{\lambda(.)\right\} = \begin{Bmatrix} \dfrac{\partial(.)}{\partial x} \\ \dfrac{\partial(.)}{\partial y} \\ \dfrac{\partial(.)}{\partial z} \end{Bmatrix}. \tag{8}$$

The displacement field within a finite element with M nodes is expressed with mode displacements and shape functions $\phi_k\left(x, y, z\right)$ specified in the local coordinate system

$$\begin{Bmatrix} h_x \\ h_y \\ h_z \end{Bmatrix} = \sum_{k=1}^{M} \begin{Bmatrix} h_{xk} \\ h_{yk} \\ h_{zk} \end{Bmatrix} \phi_k , \tag{9}$$

where k is the ordinary number of a node. Based on (4), the relation (7) is also valid for the modes specified in the global coordinate system

$$\begin{Bmatrix} H_x \\ H_y \\ H_z \end{Bmatrix} = \sum_{k=1}^{M} \begin{Bmatrix} H_{xk} \\ H_{yk} \\ H_{zk} \end{Bmatrix} \phi_k . \tag{10}$$

The stress tensor is determined in the local coordinate system, σ_{pq}, according to continuum mechanics [4]:

$$\Sigma_{kl} = c_{kp} c_{lq} \sigma_{pq}, \qquad k, l, p, q = 1, 2, 3. \tag{11}$$

It is symmetric and can also be presented in the matrix notation:

$$\left[\Sigma\right] = \left[c\right]\left[\sigma\right]\left[c\right]^T . \tag{12}$$

By using the above relations, Eq. (2) for a volume finite element yields

$$k_{ij}^{Ge} = \sum_{k=1}^{M}\sum_{l=1}^{M}\iiint_{v} H_{kl}^{ij}\left\langle \lambda\left(\phi_k\right)\right\rangle [S]\left\{\lambda\left(\phi_k\right)\right\}\mathrm{d}v\,, \tag{13}$$

where

$$H_{kl}^{ij} = H_{xk}^{i}H_{xl}^{j} + H_{yk}^{i}H_{yl}^{j} + H_{zk}^{i}H_{zl}^{j}, \tag{14}$$

$$[S] = [c][\Sigma][c]^T\,. \tag{15}$$

The hull of a ship is a thin-walled structure comprised of reinforced panels, i.e. plates and beams. Shell finite elements with 6 d.o.f. per node with corresponding shape functions are used for the approximation of displacement field within node translations and rotations. The conventional stiffness matrix has to be determined with these shape functions so that membrane and bending stresses can be calculated later on. However, the mass matrix as well as the geometric stiffness matrix can also be derived by employing simpler shape functions related only to translation.

Let us consider a flat shell element in the x, y plane. The simplification of ignoring rotational degrees of freedom offers an advantage in the shape function $\phi_k\left(x,y\right)$ being the same for the element strain and deflection. In that case, Eq. (13) for a volume element is directly applicable for determining the geometric stiffness matrix of a shell element:

$$k_{ij}^{Ge} = t\sum_{k=1}^{M}\sum_{l=1}^{M}H_{kl}^{ij}\iint_{A}\left\langle \lambda\left(\phi_k\right)\right\rangle [S]\left\{\lambda\left(\phi_k\right)\right\}\mathrm{d}A\,, \tag{16}$$

where t and A are the shell thickness and area, respectively, and

$$\left\{\lambda\left(\phi_k\right)\right\} = \begin{Bmatrix} \dfrac{\partial \phi_k}{\partial x} \\ \dfrac{\partial \phi_k}{\partial y} \\ 0 \end{Bmatrix}. \tag{17}$$

The effect of the simplified modal geometric stiffness matrix on the accuracy of results is investigated in the following sections.

3. Mass modelling in the BEAM vibration analysis

The analysis of flexural natural vibrations of a free beam is done by solving the eigenvalue problem by means of an in-house code written in [5]. The governing equation of beam flexural natural vibrations is [6]:

$$\left(\mathbf{K} - \omega^2\mathbf{M}\right)\boldsymbol{\delta} = \mathbf{0}\,, \tag{18}$$

where $\mathbf{K}$ is the global stiffness matrix, $\mathbf{M}$ is the global mass matrix, $\boldsymbol{\delta}$ represents the displacement vector and ω is the natural frequency. The solution of the governing equation, i.e. eigenvalues and eigenvectors which represent natural frequencies and natural modes, respectively, is obtained from the condition

$$\mathrm{Det}\left(\mathbf{K} - \omega^2\mathbf{M}\right) = 0\,. \tag{19}$$

The finite element stiffness and mass matrices derived with the shape functions in the form of the third order (Hermitian) polynomials, yield [6]:

$$\mathbf{k} = \frac{2EI}{l^3}\begin{bmatrix} 6 & 3l & -6 & 3l \\ & 2l^2 & -3l & l^2 \\ & & 6 & -3l \\ Symm. & & & 2l^2 \end{bmatrix} \tag{20}$$

$$\mathbf{m}_1 = \frac{ml}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ & 4l^2 & 13l & -3l^2 \\ & & 156 & -22l \\ Symm. & & & 4l^2 \end{bmatrix}, \tag{21}$$

where l is the element length and m is the mass per unit length.

The consistent mass matrix can be determined in a simpler way by the first order polynomial shape functions related only to the beam deflection. In that case one obtains

$$\mathbf{m}_2 = \frac{ml}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{22}$$

The third possibility is to directly use lumped masses, i.e.

$$\mathbf{m}_3 = \frac{ml}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{23}$$

For diagonal angular terms in (22) and (23), a very small positive value has to be assumed in order to ensure positive definite matrices and a possibility of calculation.

The application of different mass modelling and the resulting accuracy is illustrated for the case of a free beam with the following properties:

Length	$L = 40$ m
Breadth	$B = 2$ m
Height	$H = 1$ m
Cross-section area	$A = 2$ m^2
Moment of inertia of cross-section	$I = 0.1667$ m^4
Mass	$M = 6.28 \cdot 10^5$ kg
Young's modulus	$E = 2.1 \cdot 10^{11}$ N/m^2

The beam is discretisized into 8 finite elements. Table 1 contains the obtained natural frequencies as well as the analytically determined values according to the formula

$$\omega_n = \frac{(\beta_n L/2)^2}{(L/2)^2}\sqrt{\frac{EI}{m}}, \tag{24}$$

where the roots of the frequency equations for the symmetric modes are the following:

$$\beta_0 L/2 = 0, \quad \beta_2 L/2 = 2.365, \quad \beta_4 L/2 = 5.497, \tag{25}$$

and for the skew (antisymmetric) modes

$$\beta_1 L/2 = 0, \quad \beta_3 L/2 = 3.925, \quad \beta_5 L/2 = 7.068. \tag{26}$$

Discrepancies in the results for different mass modelling with respect to the analytical solution are also included in Table 1. The level of accuracy of the $\mathbf{m}_2$ specification is lower at the higher modes than that of $\mathbf{m}_1$, but for the first mode, which is very important in ship hydroelasticity, it is acceptable. So, it seems rational to investigate the influence of the ignored rotational d.o.f. in the geometric stiffness matrix on buckling analysis results.

Since the $\mathbf{m}_2$ and the $\mathbf{m}_3$ mass formulation overestimates and underestimates the results, respectively, one can use the hybrid mass matrix $\mathbf{m}_{23} = (\mathbf{m}_2 + \mathbf{m}_3)/2$. Thus, the discrepancies are considerably reduced, Table 1.

Table 1 Natural frequencies of beam flexural vibrations ωi [Hz], 8 finite elements

Mode no.	Consistent mass $\mathbf{m}_1$	Simplified mass $\mathbf{m}_2$	Lumped mass $\mathbf{m}_3$	Hybrid mass $\mathbf{m}_{23}$	Analytical solution	Discrepancy			
						δ_1 (%)	δ_2 (%)	δ_3 (%)	δ_{23} (%)
1	3.323	3.374	3.171	3.268	3.323	0	1.51	-4.79	-1.70
2	9.165	9.687	8.481	9.025	9.151	0.15	5.53	-7.90	-1.40
3	17.994	20.149	16.180	17.834	17.951	0.24	10.90	-10.95	-0.66
4	29.841	35.746	26.079	29.749	29.678	0.55	16.98	-13.80	0.24

The natural vibration analysis of the considered beam has also been performed using commercial packages SESAM [9] and NASTRAN [10], by taking into account consistent and lumped mass distribution as well as the coupled mass matrix $\mathbf{m}_{13} = (\mathbf{m}_1 + \mathbf{m}_3)/2$. The superior behaviour of mass matrices computed from the averaged consistent and lumped mass matrix is shown in [11]. The beam is discretisized in the same way as in the previous case. The obtained results are listed in Table 2. Application of the coupled mass increases accuracy only slightly with respect to the lumped mass, and it is not as effective as in the case of longitudinal vibrations, [10]. Much better results are obtained with the hybrid matrix $\mathbf{m}_{23}$, Table 1. The first two natural modes obtained by both SESAM and NASTRAN are shown in Figures 1 and 2, respectively.

Table 2 Natural frequencies of beam flexural vibrations ω_i [Hz], SESAM and NASTRAN, 8 finite elements

Mode no.	SESAM				NASTRAN	
	Consistent mass, $\mathbf{m}_1$	Lumped mass, $\mathbf{m}_3$	Discrepancy δ_1 (%)	Discrepancy δ_3 (%)	Coupled mass, $\mathbf{m}_{13}$	Discrepancy δ_{13} (%)
1	3.319	3.123	-0.12	-6.40	3.168	-4.89
2	9.125	8.239	-0.28	-11.07	8.451	-8.28
3	17.832	15.338	-0.67	-17.04	16.052	-11.83
4	29.394	24.224	-0.97	-22.51	25.692	-15.51

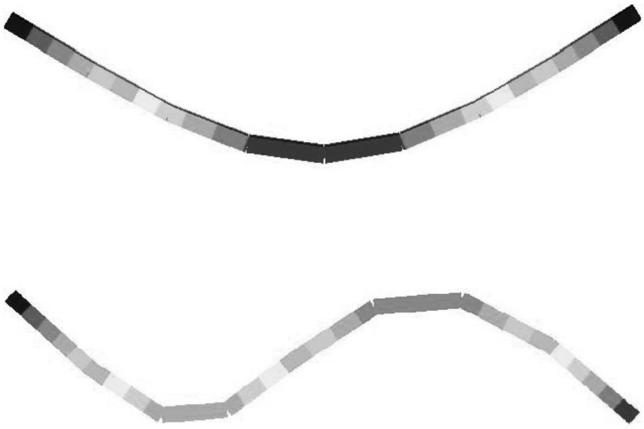

Fig. 1 The first and the second natural mode of beam vertical vibrations, SESAM

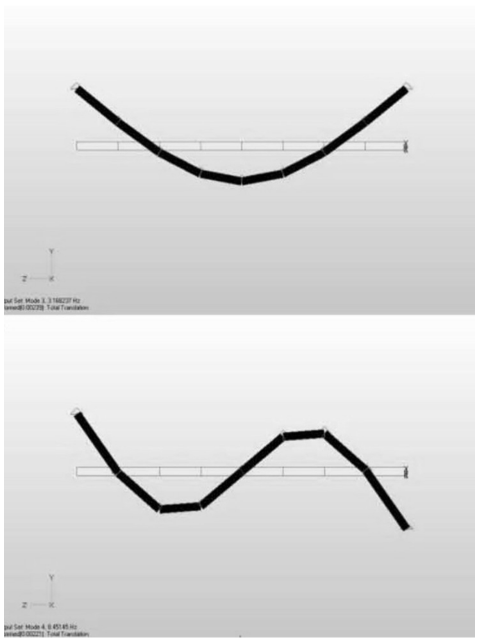

Fig. 2 The first and the second natural mode of beam vertical vibrations, NASTRAN

In order to demonstrate the convergence of the numerically determined results to the analytical solution, the calculation of beam vibrations is repeated by taking 16 finite elements into account, Table 3. When comparing the values of discrepancies in Table 3 with those in Table 2, it is obvious that δ_3 and δ_{13} are considerably reduced. The values of δ_1 are slightly increased but are still very small.

Table 3 Natural frequencies of beam flexural vibrations ωi [Hz], SESAM and NASTRAN, 16 finite elements

Mode no.	SESAM				NASTRAN	
	Consistent mass, $\mathbf{m}_1$	Lumped mass, $\mathbf{m}_3$	Discrepancy δ_1 (%)	Discrepancy δ_3 (%)	Coupled mass, $\mathbf{m}_{13}$	Discrepancy δ_{13} (%)
1	3.319	3.231	-0.12	-2.84	3.280	-1.31
2	9.118	8.696	-0.36	-5.23	8.944	-2.31
3	17.786	16.416	-0.93	-9.35	17.324	-3.62
4	29.211	26.240	-1.60	-13.10	28.248	-5.06

4. Modelling of geometric stiffness in BEAM BUCKLING

The governing equation of beam buckling yields [6]:

$$\left(\mathbf{K}-\eta\mathbf{K}_G\right)\delta=\mathbf{0}\,, \tag{27}$$

where $\mathbf{K}_G$ is the global geometric stiffness matrix. The geometric stiffness matrix of finite element depends on the external compression load (axial force) N, Figure 3, and for the case of its constant value along the element and the third order polynomial shape functions, it is given in the following form [6]:

$$\mathbf{k}_{G1}=\frac{N}{30l}\begin{bmatrix}36 & 3l & -36 & 3l\\ & 4l^2 & -3l & -l^2\\ & & 36 & -3l\\ Symm. & & & 4l^2\end{bmatrix}. \tag{28}$$

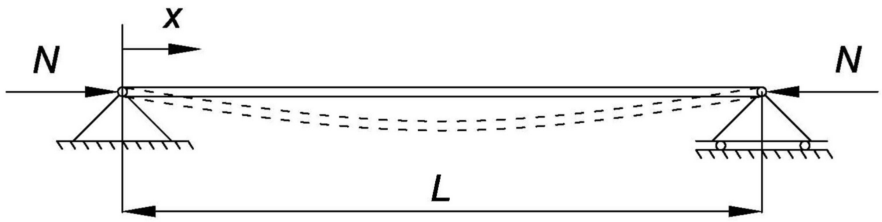

Fig. 3 Beam buckling

The geometric stiffness matrix can be simplified by ignoring the rotational d.o.f. that is achieved by applying the first order polynomial shape functions:

$$\mathbf{k}_{G2}=\frac{N}{l}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}. \tag{29}$$

Zero diagonal terms related to rotations have to be replaced with some positive small terms in order to ensure positive definite matrices and numerical stability. By solving Eq. (27), one obtains the value of η that represents the ratio of the critical force and the imposed axial force. First, Eq. (27) is solved with the consistent geometric stiffness matrix $\mathbf{k}_{G1}$, Eq. (28), and then with the simplified one, $\mathbf{k}_{G2}$, Eq. (29).

The value of η can also be determined as the ratio of critical and imposed load $\frac{N_n}{N}$, where the critical force is obtained by the analytical formula for different sinusoidal buckling modes [12]:

$$N_n=\frac{n^2\pi^2 EI}{L^2}, \tag{30}$$

where n is the ordinary mode number. In the buckling analysis, all input data are the same as in the previous section, and the value of the assumed axial force N is 10 kN.

The results obtained by two different approaches, $\mathbf{k}_{G1}$ and $\mathbf{k}_{G2}$, and 8 finite elements are presented in Table 4. The discrepancy of the former is negligible for the first buckling mode, while that of the latter is 1.4%, which is acceptable from the engineering point of view. As expected, similar to the case of natural vibrations, discrepancies become larger at the higher modes. They are considerably reduced by the double finite element mesh refinement, Table 5.

Table 4 Beam buckling factor η, 8 finite elements

Mode no.	Consistent $\mathbf{k}_{G1}$	Simplified $\mathbf{k}_{G2}$	Analytical solution	Discrepancies	
				δ_1 (%)	δ_2 (%)
1	21591	21869	21570	0.09	1.37
2	86403	90883	86280	0.14	5.06
3	194793	217631	194130	0.34	10.80
4	348035	420000	345120	0.84	17.82

Table 5 Beam buckling factor η, 16 finite elements

Mode no.	Consistent $\mathbf{k}_{G1}$	Simplified $\mathbf{k}_{G2}$	Analytical solution	Discrepancies	
				δ_1 (%)	δ_2 (%)
1	21589	21659	21570	0.088	0.412
2	86363	87474	86280	0.095	1.365
3	194340	199988	194130	0.108	2.929
4	345613	363532	345120	0.143	5.065

5. Conclusion

The finite element formulation of the unified restoring and the geometric stiffness in the hydroelastic analysis of ship and offshore structures is a rather complex task. By ignoring rotational terms in the geometric stiffness matrix, while keeping the necessary accuracy, the problem is simplified. Hence, the same shape functions as in the case of the in-plane plate deformation can be used for its deflection. This simplification should not affect the results significantly, as shown in the case of free vibration analysis. The governing equations for both problems are similar, but instead of the mass matrix, the geometric stiffness matrix is present in the buckling problem. However, it is more difficult to explain the physical background of particular terms in the geometric stiffness matrix than of those in the mass matrix. Because of that, free vibration analysis has been done first and it has shown that natural frequencies of the first natural mode obtained by different modelling are in good agreement. After that, the extension of the investigation to beam buckling resulted in even better agreement. Although the performed calculations are rather simple, the conclusion that rotations can sometimes be omitted is quite important because it makes the formulation of more complex issues (for example, finite element formulation of the unified restoring stiffness and the geometric stiffness in hydroelastic analysis) easier. Actually, the accuracy depends on the finite element mesh density. Hence, it is necessary to have at least 10 finite elements between two vibration nodes of the highest included natural mode.

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