SCHUR-CONVEXITY OF ČEBIŠEV FUNCTIONAL

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Abstract. In this paper the Čebišev weighted functional $T(p; f, g; a, b)$ is regarded as a function of two variables

$$T(p; f, g; x, y) = \frac{\int_x^y p(t)f(t)g(t)dt}{\int_x^y p(t)dt} - \left(\frac{\int_x^y p(t)f(t)dt}{\int_x^y p(t)dt}\right)\left(\frac{\int_x^y p(t)g(t)dt}{\int_x^y p(t)dt}\right),\ (x, y) \in [a, b]\times[a, b]$$

where f, g and $p > 0$ are Lebesgue integrable functions. The property of Schur-covexity (Schur-concavity) of this function is proved.

1. Introduction

Let I be an interval with nonempty interior and $\mathbf{x} = (x_i, x_2, .., x_n)$ and $\mathbf{y} = (y_i, y_2, .., y_n)$ in I^n be two n-tuples such that $\mathbf{x} \prec \mathbf{y}$, i.e.

$$\begin{aligned} \sum_{i=1}^{k} x_{[i]} &\leq \sum_{i=1}^{k} y_{[i]}, \quad k = 1, ..., n-1 \\ \sum_{i=1}^{n} x_{[i]} &= \sum_{i=1}^{n} y_{[i]}, \end{aligned}$$

where $x_{[i]}$ denotes the i th largest component in x.

Definition 1. *Function $F : I^n \to \mathbb{R}$ is Schur-convex on I^n if*

$$F(x_i, x_2, .., x_n) \leq F(y_i, y_2, .., y_n)$$

for each two n-tuples $\mathbf{x}$ and $\mathbf{y}$ such that it holds $\mathbf{x} \prec \mathbf{y}$ on I^n.
Function F is Schur-concave on I^n if and only if $-F$ is Schur-convex.

The next lemma gives us a necessery and sufficient condition for verifying the Schur-convexity property of F when $n = 2$ ([4, p. 333], [3, p.57]).

Lemma A 1. *Let $F : I^2 \to \mathbb{R}$ be a continuous function on I^2 and differentiable in interior of I^2. Then F is Schur-convex (Schur-concave) if and only if it is symmetric and*

$$(1) \qquad \left(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\right)(y - x) \geq 0$$

holds (reverses) for all $x, y \in I$, $x \neq y$.

The authors in [2] were inspired by some inequalities concerning gamma and digamma function and proved the following result for the integral arithmetic mean:

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Theorem A 1. *Let f be a continuous function on I. Then*

$$\begin{aligned} F(x,y) &= \frac{1}{y-x}\int_x^y f(t)dt \\ F(x,x) &= f(x) \end{aligned}$$

is Schur-convex (Schur-concave) on I^2 if and only if f is convex (concave) on I.

Also, in [2], applications to logarithmic mean are given.

The authors in [5] proved the Schur convexity of the weighted integral arithmetic mean of function f :

Theorem A 2. *Let f be a continuous function on I, let p be a positive continuous weight on I. Then*

$$F_p(x,y) = \begin{cases} \frac{1}{\int_x^y p(t)dt}\int_x^y p(t)f(t)dt, & x,y\in I, x\neq y \\ f(x), & x=y \end{cases}$$

is Schur-convex (Schur-concave) on I^2 if and only if the inequality

$$\frac{\int_x^y p(t)f(t)dt}{\int_x^y p(t)dt} \leq \frac{p(x)f(x)+p(y)f(y)}{p(x)+p(y)}$$

holds (reverses) for all x,y in I.

The Čebišev functional $T(f,g;a,b)$ is defined for two Lebesgue integrable f and g on interval $[a,b]\in\mathbb{R}$ as

$$T(f,g;a,b) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - (\frac{1}{b-a}\int_a^b f(t)dt)(\frac{1}{b-a}\int_a^b g(t)dt).$$

Because the Čebišev functional can be express in the term of the integral arithmetic mean, we were inspired by above results in Theorem A and in [1] we generalized these results by proving the Schur-convexity of function

$$T(f,g;x,y) = \frac{1}{y-x}\int_x^y f(t)g(t)dt - (\frac{1}{y-x}\int_x^y f(t)dt)(\frac{1}{y-x}\int_x^y g(t)dt),$$

$(x,y)\in[a,b]\times[a,b]$.

Theorem A 3. *Let f and g be Lebesgue integrable functions on $I=[a,b]$. If they are monotone in the same sense (in the opposite sense) then $T(x,y) := T(f,g;x,y)$, $(x,y)\in[a,b]\times[a,b]\in\mathbb{R}^2$ is Schur-convex (Schur-concave) on $[a,b]\times[a,b]$.*

We used the well-known Čebišev inequality:

Theorem A 4. *Let f and g be Lebesgue integrable on interval $[a,b]$. If f and g are monotonic in the same sense (in the opposite sense) then*

$$T(f,g;a,b)\geq 0(\leq 0).$$

In this paper we will consider weighted Čebišev functional defined as

$$T(p;f,g;x,y) = \frac{\int_x^y p(t)f(t)g(t)dt}{\int_x^y p(t)dt} - (\frac{\int_x^y p(t)f(t)dt}{\int_x^y p(t)dt})(\frac{\int_x^y p(t)g(t)dt}{\int_x^y p(t)dt}),\ (x,y)\in[a,b]\times[a,b].$$

for f and g Lebesgue integrable functions on $I=[a,b]$ and p a positive continuous weight on I such that pf and pg are also Lebesgue integrable functions on I.

Let we use the following notations

$P(x,y) := \int_x^y p(t)dt$,
$\overline{f_p}(x,y) := \frac{1}{\int_x^y p(t)dt}\int_x^y p(t)f(t)dt$ and $\overline{g}_p(x,y) := \frac{1}{\int_x^y p(t)dt}\int_x^y p(t)g(t)dt$.
So, the $T(p;f,g;x,y)$ can be rewritten as

$$T(p;f,g;x,y) = \frac{\int_x^y p(t)f(t)g(t)dt}{P(x,y)} - \overline{f_p}(x,y)\cdot\overline{g}_p(x,y),\ (x,y)\in[a,b]\times[a,b].$$

In this paper we obtain corresponding result to Theorem A2 for weighted Čebišev functional and show the another proof of Theorem A3.

2. Results

Theorem 2.1. *Let f and g be Lebesgue integrable functions on $I=[a,b]$ and let p be a positive continuous weight on I such that pf and pg are also Lebesgue integrable functions on $I=[a,b]$. Then $T(p;x,y) := T(p;f,g;x,y)$, is Schur-convex (Schur-concave) on $I^2=[a,b]\times[a,b]$ if and only if the inequality*
(2)
$$T(p;x,y) \le \frac{p(x)(\overline{f_p}(x,y)-f(x))(\overline{g_p}(x,y)-g(x)) + p(y)(\overline{f_p}(x,y)-f(y))(\overline{f_g}(x,y)-g(y))}{p(x)+p(y)}.$$

holds (reverses) for all x,y in I.

*Proof.*To prove the Schur-convexity of $T(p;x,y)$ by Lemma A1 the inequality (1) it is sufficient to prove $(\frac{\partial T(p;x,y)}{\partial y} - \frac{\partial T(p;x,y)}{\partial x})(y-x)\ge 0$, for all $x,y\in[a,b]$, since the function $T(p;x,y) := T(p;f,g;x,y)$ is evidently symmetric.

Now, we calculate $\frac{\partial T(p;x,y)}{\partial y}$ and $\frac{\partial T(p;x,y)}{\partial x}$:

$$\begin{aligned}\frac{\partial T(p;x,y)}{\partial x} &= \frac{p(x)}{P(x,y)}[T(p;x,y) - (\overline{f_p} - f(x))(\overline{g}_p - g(x))];\\ \frac{\partial T(p;x,y)}{\partial y} &= \frac{p(y)}{P(x,y)}[-T(p;x,y) + (\overline{f_p} - f(y))(\overline{g}_p - g(y))].\end{aligned}$$

Direct calculation yields that

$$\begin{aligned}&(\frac{\partial T(p;x,y)}{\partial y} - \frac{\partial T(p;x,y)}{\partial x})(y-x)\\ =\ & \frac{[p(x)+p(y)]}{P(x,y)}\\ &\{-T(p;x,y) + \frac{p(x)(\overline{f_p} - f(x))(\overline{g}_p - g(x)) + p(y)(\overline{f_p} - f(y))(\overline{g}_p - g(y))}{p(x)+p(y)}\}(y-x).\end{aligned}$$

Since $\frac{y-x}{P(x,y)}\ge 0$, then a necessary and sufficient condition for Schur-convexity of $T(p;x,y)$ is that holds

$$T(p;x,y) \le \frac{p(x)(\overline{f_p} - f(x))(\overline{g}_p - g(x)) + p(y)(\overline{f_p} - f(y))(\overline{g}_p - g(y))}{p(x)+p(y)}.$$

Similarly, we conclude the Schur-concavity of $T(p;x,y)$.

For special choice $p(t) = 1, t \in [a, b]$ and using the short notation for the integral means: $\overline{f}(x, y) := \frac{1}{y-x} \int_x^y g(t)dt$ and $\overline{g}(x, y) := \frac{1}{y-x} \int_x^y g(t)dt$. in Theorem 2.1 we can obtain the following result:

Corollary 2.1. *Let f and g be Lebesgue integrable functions on $I = [a, b]$. Then $T(x, y) := T(f, g; x, y)$, is Schur-convex (Schur-concave) on $I^2 = [a, b] \times [a, b]$ if and only if the inequality*

$$T(x,y) \leq \frac{1}{2}(\overline{f}(x,y) - f(x))(\overline{g}(x,y) - g(x)) + (\overline{f}(x,y) - f(y))(\overline{f}(x,y) - g(y)) \tag{3}$$

holds (reverses) for all x, y in I.

Remark 2.1. *Using Theorem 2.1 for special choice $p(t) = 1, t \in [a, b]$ i.e. Corrolary 2.1 we can obtain result in Theorem A3 according conditions that functions f and g are monotone in the same sense (in the opposite sense).*

Another proof of Theorem A3:

There are three cases to be considered according monotonicity of functions.

Case 1. Let f and g be two increasing functions on $[a, b]$ and $x < y$. So, we have $f(x) \leq f(t) \leq f(y)$ and $g(x) \leq g(t) \leq g(y)$ and it yields

$$(f(y) - f(t))(f(t) - f(x)) \geq 0, \tag{4}$$
$$(g(y) - g(t))(g(t) - g(x)) \geq 0. \tag{5}$$

In the proof in [1] *we showed that then the inequality (3) holds.*

So, Corrolary 2.1 implies the property of Schur-convexity of $T(f, g; x, y)$.

We have to remark that for $x > y$ the inequalities in (4) and (5) still are valid and we can find that

$$T(f,g;y,x) \leq \frac{1}{2}[(\overline{f}(x,y) - f(y))(\overline{g}(x,y) - g(y)) + (f(x) - \overline{f}(x,y))(g(x) - \overline{g}(x,y))].$$

As the T is symmetric, it is obvious that

$$\begin{aligned} T(f,g;x,y) &\leq \frac{1}{2}[(\overline{f}(x,y) - f(y))(\overline{g}(x,y) - g(y)) + (f(x) - \overline{f}(x,y))(g(x) - \overline{g}(x,y))] \\ &= \frac{1}{2}[(\overline{f}(x,y) - f(x))(\overline{g}(x,y) - g(x)) + (f(y) - \overline{f}(x,y))(g(y) - \overline{g}(x,y))]. \end{aligned}$$

Again, the inequality (3) holds and Corollary 2.1 implies Shour-convexity of $T(f, g; x, y)$.

Simillary as in [1] *for Case 2. we suppose that f and g are both decreasing functions on $[a, b]$ and $x < y$. Since $f(x) \geq f(t) \geq f(y)$ and $g(x) \geq g(t) \geq g(y)$ the inequalities in (4) and (5) again are valid and the proof is the same as in Case 1.*

Case 3. Let f be an increasing function and g decreasing function. Note that we can consider Case 1. for function f and $-g$ and in [1] *we proved reverse inequality in (3) for functions f and g. Similarly as in Case 1., according Corrolary 2.1 reverse inequality (3) implies the Schur-concavity of $T(f, g; x, y)$.*

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