SCHUR-CONVEXITY OF THE WEIGHTED ČEBYŠEV FUNCTIONAL II

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Abstract. In this paper the weighted Čebyšev functional $T(p; f, g; a, b)$ is regarded as a function of two variables

$$T(p; f, g; x, y) = \frac{\int_x^y p(t)f(t)g(t)dt}{\int_x^y p(t)dt} - \left(\frac{\int_x^y p(t)f(t)dt}{\int_x^y p(t)dt}\right)\left(\frac{\int_x^y p(t)g(t)dt}{\int_x^y p(t)dt}\right), \ (x, y) \in [a, b] \times [a, b]$$

where f, g and $p > 0$ are Lebesgue integrable functions. For a function

$$K(p; f, g; x, y) = \left(\int_x^y p(t)dt\right)^2 T(p; f, g; x, y) \ (x, y) \in [a, b] \times [a, b]$$

the property of Schur-covexity, Schur-geometric convexity, Schur-harmonic convexity and $(1, 1)$-convexity is proved.

1. Introduction

Let f, g and $p > 0$ be Lebesgue integrable functions on the interval $I = [a, b] \subseteq \mathbb{R}$. In this paper the weighted Čebyšev functional $T(p; f, g; a, b)$ is regarded as a function of two variables

$$T(p; f, g; x, y) = \frac{\int_x^y p(t)f(t)g(t)dt}{\int_x^y p(t)dt} - \left(\frac{\int_x^y p(t)f(t)dt}{\int_x^y p(t)dt}\right)\left(\frac{\int_x^y p(t)g(t)dt}{\int_x^y p(t)dt}\right), \ (x, y) \in I^2.$$

In [4] we proved Schur-convexity of a function $T(1; f, g; x, y)$ with $(x, y) \in I^2$.

Theorem A 1. *Let f and g be Lebesgue integrable functions on $I = [a, b] \subseteq \mathbb{R}$. If they are monotone in the same sense (in the opposite sense) then $T(x, y) := T(1; f, g; x, y)$, $(x, y) \in I^2$ is Schur-convex (Schur-concave) on I.*

Using the following notations:
$P(x, y) := \int_x^y p(t)dt$,
$\overline{f_p}(x, y) := \frac{1}{\int_x^y p(t)dt} \int_x^y p(t)f(t)dt$ and $\overline{g}_p(x, y) := \frac{1}{\int_x^y p(t)dt} \int_x^y p(t)g(t)dt$
we obtained next result for the weighted Čebyšev functional (see [5]):

Theorem A 2. *Let f and g be Lebesgue integrable functions on $I = [a, b] \subseteq \mathbb{R}$ and let p be a positive continuous weight on I such that pf and pg are also Lebesgue integrable functions on I. Then $T(x, y) := T(p; f, g; x, y)$ is Schur-convex (Schur-concave) on I^2 if and only if the inequality*

(1)
$$T(x, y) \leq \frac{p(x)(\overline{f_p}(x, y) - f(x))(\overline{g_p}(x, y) - g(x)) + p(y)(\overline{f_p}(x, y) - f(y))(\overline{g_p}(x, y) - g(y))}{p(x) + p(y)}.$$

holds (reverses) for all x, y in I.

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For a function $K : I^2 \subseteq \mathbb{R}^2 \to \mathbb{R}$ defined by

$$K(p; f, g; x, y) = \left(\int_x^y p(t)dt \right)^2 \cdot T(p; f, g; x, y), \ (x, y) \in I^2 \tag{2}$$

the author in [17] proved the following statement:

Theorem A 3. *Let $f, g : [a, b] \to \mathbb{R}$ be Lebesgue integrable functions, and let $p : [a, b] \to \mathbb{R}_+$ be a Lebesgue integrable function. A function $K(x, y) := K(p; f, g; x, y)$ defined as (2) is increasing (decreasing) with y on $I = [a, b]$ and decreasing (increasing) with x on I if f and g are monotone in the same sense (in the opposite sense).*

In this paper we prove the property of Schur-convexity, Schur-geometric convexity and Schur-harmonic convexity of a function $K(x, y)$ with (x, y), depending of monotonicity and simultan ordering of the functions f and g. We also show $(1, 1)$-convexity of a function K.

2. Definitions and proprties

The concepts of majorizations and Schur-convex functions involve convex functions and measure of the diversity of the components of an n-tuple in $\mathbb{R}^n$. Most of the basic results are given in Marshall and Olkin's book [8]. In the recently references [1], [2], [3], [10], [11], [13], [15], [16], we can find the definitions and applications of the Schur-convex, Schur-geometrically convex and Schur-harmonic convex functions.

In this section we will recall usefull definitions, lemmas and theorems:

Definition 1. *Let $\mathbf{x}, \mathbf{y}$ be in $E \subseteq \mathbb{R}^n$ and let $x_{[i]}$, $y_{[i]}$ denote the i th largest component in $\mathbf{x}$ and $\mathbf{y}$. We say $\mathbf{y}$ majorizes $\mathbf{x}$, denote $\mathbf{x} \prec \mathbf{y}$ if*

$$\begin{aligned} \sum_{i=1}^{k} x_{[i]} &\leq \sum_{i=1}^{k} y_{[i]}, \quad k = 1, ..., n-1, \\ \sum_{i=1}^{n} x_{[i]} &= \sum_{i=1}^{n} y_{[i]}. \end{aligned}$$

Definition 2. *Let $\mathbf{x}, \mathbf{y}$ be in $E \subseteq \mathbb{R}^n$. A function $F : E \to \mathbb{R}$ is called a Schur-convex function on E if*

$$F(x_1, x_2, .., x_n) \leq F(y_1, y_2, .., y_n)$$

for each $\mathbf{x}$ and $\mathbf{y}$ in E such that $\mathbf{x} \prec \mathbf{y}$.

A function F is Schur-concave if and only if $-F$ is a Schur-convex function.

Definition 3. *Let $\mathbf{x}, \mathbf{y}$ be in $E \subseteq \mathbb{R}_+^n$. A function $F : E \to [0, \infty)$ is called a Schur-geometrically convex function on E if*

$$F(x_1, x_2, .., x_n) \leq F(y_1, y_2, .., y_n)$$

for each two positive $\mathbf{x}$ and $\mathbf{y}$ in E such that $(\ln x_1, \ln x_2, .., \ln x_n) \prec (\ln y_1, \ln y_2, .., \ln y_n)$ i.e $\mathbf{y}$ logarithm majorizes $\mathbf{x}$.

A function F is Schur-geometrically concave if and only if $-F$ is a Schur-geometrically convex function.

Definition 4. *Let* $\mathbf{x}, \mathbf{y}$ *be in* $E \subseteq \mathbb{R}_+^n$. *A function* $F : E \to [0, \infty)$ *is called a Schur-harmonic convex function on* E *if*

$$F(x_1, x_2, .., x_n) \leq F(y_1, y_2, .., y_n)$$

for each two positive $\mathbf{x}$ *and* $\mathbf{y}$ *on* E *such that it holds* $(\frac{1}{x_1}, \frac{1}{x_2}, .., \frac{1}{x_n}) \prec (\frac{1}{y_1}, \frac{1}{y_2}, .., \frac{1}{y_n})$.

A function F *is Schur-harmonic concave if and only if* $-F$ *is a Schur-harmonic convex function.*

The next lemmas gives us the characterisations of Schur-convexity, Schur-geometrically convexity and Schur-harmonic convexity (see [8, p.57], [11, p.333], [14], [15, p.108], [16]):

Lemma A 1. *Let* $E \subseteq \mathbb{R}^n$ *be a symmetric convex set with a nonempty interior. Let* $F : E \to \mathbb{R}$ *be a continuous function on* E *and differentiable on the interior of* E. *Then* F *is Schur-convex (Schur-concave) if and only if it is symmetric and*

$$\left(\frac{\partial F}{\partial x_2} - \frac{\partial F}{\partial x_1}\right)(x_2 - x_1) \geq 0 \ (\leq 0) \tag{3}$$

holds for all $\mathbf{x}$ *in the interior of* E, $x_1 \neq x_2$.

Lemma A 2. *Let* $E \subseteq \mathbb{R}_+^n$ *be a symmetric logarithm convex set with a nonempty interior i.e.* $\ln E = \{\ln \mathbf{x} = (\ln x_1, ..., \ln x_n) : \mathbf{x} \in E\}$ *is a convex set. Let* $F : E \to [0, \infty)$ *be a continuous function on* E *and differentiable on the interior of* E. *Then* F *is Schur-geometrically convex (Schur-geometrically concave) if it is symmetric and the inequality*

$$\left(x_2 \frac{\partial F}{\partial x_2} - x_1 \frac{\partial F}{\partial x_1}\right)(\ln x_2 - \ln x_1) \geq 0 \ (\leq 0) \tag{4}$$

holds for all $\mathbf{x}$ *in the interior of* E, $x_1 \neq x_2$.

Lemma A 3. *Let* $E \subseteq \mathbb{R}_+^n$ *be a symmetric harmonic convex set with a nonempty interior.i.e.* $\mathbf{1}/E = \{1/\mathbf{x} = (\frac{1}{\mathbf{x}_1}, ..., \frac{1}{\mathbf{x}_n}) : \mathbf{x} \in E\}$ *is a convex set. Let* $F : E \to [0, \infty)$ *be a continuous function on* E *and differentiable on the interior of* E. *Then* F *is Schur-harmonic convex (Schur-harmonic concave) if it is symmetric and*

$$\left(x_2^2 \frac{\partial F}{\partial x_2} - x_1^2 \frac{\partial F}{\partial x_1}\right)(x_2 - x_1) \geq 0 \ (\leq 0) \tag{5}$$

holds for all $\mathbf{x}$ *in the interior of* E, $x_1 \neq x_2$.

Definition 5. *The functions* f *and* $g : I^n \to \mathbb{R}$ *are similarly ordered if*

$$(f(x_1, x_2, .., x_n) - f(y_1, y_{2,}.., y_n)) \cdot (g(x_1, x_2, ., x_n) - g(y_1, y_{2,}.., y_n)) \geq 0,$$

for each two n*-tuples* $\mathbf{x}$ *and* $\mathbf{y}$ *on* I^n.

Function f *and* g *are oppositely ordered if* f *and* $-g$ *are similarly ordered.*

We recall the well-known Čebyšev inequality for monotone functions (see [9, p. 239], [11, p.197]) and for similar ordered functions (see [7, p.168], [9, p.252]):

Theorem A 4. *Let* f *and* g *be Lebesgue integrable on an interval* $I = [a, b] \subseteq \mathbb{R}$ *and let* p *be a positive continuous weight on* I *such that* pf *and* pg *are also Lebesgue integrable functions on* I. *If* f *and* g *are monotone in the same sense (in the opposite sense) then it holds*

$$T(p; f, g; a, b) \geq 0 \ (\leq 0).$$

Theorem A 5. *Let f and g be Lebesgue integrable functions on $I = [a, b] \subseteq \mathbb{R}$ and let p be a positive continuous weight on I such that pfg, pf and pg are also Lebesgue integrable functions on I. If f and g are similarly (oppositely) ordered then it holds*

$$T(p; f, g; a, b) \geq 0 \ (\leq 0).$$

Popoviciu in ([12, p.60]) used the (n, m)- divided difference of the function in the definition of the $(n, m)-$convexity (concavity) (see also [11, p.18]):

Definition 6. *A function $F : I^2 \to \mathbb{R}$ is (n, m)-convex (concave) if for all distinct points $x_0, x_1, .., x_n \in I$ and $y_0, y_1, .., y_m \in I$ yilds*

$$\left[\begin{array}{ccccc} x_0, & x_1, & ., & ., & x_n \\ y_0, & y_1, & ., & ., & y_m \end{array} \right] F = \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{F(x_i, x_j)}{\omega'(x_i) \cdot w'(y_j)} \geq 0 \ (\leq 0),$$

where $\omega(x) = \prod_{i=0}^{n} (x - x_i)$, $w(y) = \prod_{j=0}^{m} (y - y_j)$.

The next lemma give us the necessary and sufficient conditions for verifying the $(n, m)-$convexity (concavity):

Lemma A 4. *If the partial derivative $\frac{\partial^{(n+m)} F}{\partial x^n \partial y^m}$ exists then $F : I^2 \to \mathbb{R}$ is (n, m)-convex (concave) if and only if*

$$\frac{\partial^{(n+m)} F}{\partial x^n \partial y^m} \geq 0 \ (\leq 0).$$

3. Results

Theorem 3.1. *Let f and g be Lebesgue integrable on interval $I = [a, b] \subseteq \mathbb{R}$. Let p be a positive continuous weight on I such that pfg, pf and pg are also Lebesgue integrable functions on I. If f and g are monotone in the same sense (in the opposite sense) then for a function $K(x, y) := K(p; f, g; x, y)$ defined by (2) holds*

(i) $K(x, y) \geq 0 \ (\leq 0)$, for $(x, y) \in I^2$;

(ii) $K(x, y)$ is Schur-convex (Schur-concave) with (x, y) on $I^2 \subseteq \mathbb{R}^2$;

(iii) $K(x, y)$ Schur-geometrical convex (Schur-geometrical concave) with (x, y) on $I^2 \subseteq \mathbb{R}^2_+$;

(iv) $K(x, y)$ is Schur-harmonic convex (Schur-harmonic concave) with (x, y) on $I^2 \subseteq \mathbb{R}^2_+$;

(v) $K(x, y) := K(p; f, g; x, y)$ is an $(1, 1)$-concave (convex) function on $I^2 \subseteq \mathbb{R}^2$.

Proof. Let f and g be monotone in the same sense (in the opposite sense). Let p be a positive continuous weight on I such that pfg, pf and pg are also Lebesgue integrable functions on $I = [a, b]$.

We may assume that $x < y$ without loss of generality.

Now, we calculate $\frac{\partial K(x,y)}{\partial y}$, $\frac{\partial K(x,y)}{\partial x}$ and $\frac{\partial^2 K(x,y)}{\partial x \partial y}$:

$$\frac{\partial K(x,y)}{\partial y} = p(y)\int_x^y p(t)(f(t)-f(y))(g(t)-g(y))dt; \tag{6}$$

$$\frac{\partial K(x,y)}{\partial x} = -p(x)\int_x^y p(t)(f(t)-f(x))(g(t)-g(x))dt; \tag{7}$$

$$\frac{\partial^2 K(x,y)}{\partial x \partial y} = -p(x)p(y)(f(y)-f(x))(g(y)-g(x)) \tag{8}$$

(i) Applying Čebyšev inequality, Theorem A4 to the function $K(p;f,g;x,y) = [P(x,y)]^2 \cdot T(p;f,g;x,y)$ we obtain that holds $K(x,y) \geq 0$ (≤ 0).

(ii) To prove Schur-convexity of $K(p;x,y)$ (or Schur-concavity) we apply Lemma A1. It is sufficient to discuss the following inequlity $\left(\frac{\partial K(x,y)}{\partial y} - \frac{\partial K(x,y)}{\partial x}\right)(y-x) \geq 0$ (≤ 0), for all $x, y \in [a,b]$, since the function $K(x,y)$ is evidently symmetric. According Theorem A3 we know that $K(x,y)$ is increasing (decreasing) with y on I and decreasing (increasing) with x on I. So, it follows Schur-convexity (Schur-concavity) of K as in the statement (ii).

(iii) The set $I^2 \subseteq \mathbb{R}_+^2$ is a symmetric logarithm convex set. By applying the condition in Lemma A2 to the function $K(x,y)$ we conclude that $\left(y\frac{\partial K(x,y)}{\partial y} - x\frac{\partial K(x,y)}{\partial x}\right) \cdot (\ln y - \ln x) \geq 0$ (≤ 0), $(x,y) \in I^2 \subseteq \mathbb{R}_+^2$, i.e. $K(x,y)$ is Schur-geometically convex (Schur - geometically concave) with (x,y) on $I^2 \subseteq \mathbb{R}_+^2$.

(iv) The set $I^2 \subseteq \mathbb{R}_+^2$ is a symmetric harmonic convex set. According Lemma A3 we conclude that $\left(y^2\frac{\partial K(x,y)}{\partial y} - x^2\frac{\partial K(x,y)}{\partial x}\right)(y-x) \geq 0$ (≤ 0), $(x,y) \in I^2 \subseteq \mathbb{R}_+^2$, i.e. $K(x,y)$ is Schur-harmonic convex (Schur-harmonic concave) with (x,y) on $I^2 \subseteq \mathbb{R}_+^2$.

(v) Since $\frac{\partial^2 K(x,y)}{\partial x \partial y} \leq 0$ (≥ 0), Lemma A4 implies that $K(x,y)$ is the $(1,1)$-concave (convex) function on $I^2 \subseteq \mathbb{R}^2$. □

Theorem 3.2. *Let f and g be Lebesgue integrable functions on $I = [a,b] \subseteq \mathbb{R}_+$ and let p be a positive continuous weight on I such that pfg, pf and pg are also Lebesgue integrable functions on $I = [a,b]$. If f and g are similarly (oppositely) ordered then for the function $K(x,y) := K(p;f,g;x,y)$ defined by (2) holds*

(i) $K(x,y)$ is increasing (decreasing) with y on I and decreasing (increasing) with x on I;

(ii) $K(x,y) \geq 0$ (≤ 0) , for $(x,y) \in I^2 \subseteq \mathbb{R}_+^2$;

(iii) $K(x,y)$ is Schur-convex (Schur-concave) with (x,y) on $I^2 \subseteq \mathbb{R}^2$;

(iv) $K(x,y)$ is Schur-geometrical convex (Schur-geometrical concave) with (x,y) on $I^2 \subseteq \mathbb{R}_+^2$;

(v) $K(x,y)$ is Schur-harmonic convex (Schur-harmonic concave) with (x,y) on $I^2 \subseteq \mathbb{R}_+^2$;

(vi) $K(x,y)$ is an $(1,1)$-concave (convex) function on $I^2 \subseteq \mathbb{R}^2$.

Proof. Let f and g are similarly ordered (oppositely ordered) on $I = [a,b] \subseteq \mathbb{R}_+$. Let p be a positive continuous weight on I such that pfg, pf and pg are also Lebesgue integrable functions on I.

(i) From (6) and (7) we have that $\frac{\partial K(x,y)}{\partial y} \geq 0$ (≤ 0) and $\frac{\partial K(x,y)}{\partial x} \leq 0$ (≥ 0) So, it holds statement (i).

(ii) $K(p;f,g;x,y)=[P(x,y)]^2\cdot T(p;f,g;x,y)$ and Čebyšev inequality, Theorem A5 implies statement (ii).

(iii) The claim (i) implises that $\left(\frac{\partial K(x,y)}{\partial y}-\frac{\partial K(x,y)}{\partial x}\right)(y-x)\geq 0\ (\leq 0)$ on $I^2$and according Lemma A3 it follows the property of Schur-convexity (Schur-concavity) of $K(x,y)$ on I^2.

(iv) Similarly, by statement (i) we conclude that $\left(y\frac{\partial K(x,y)}{\partial y}-x\frac{\partial K(x,y)}{\partial x}\right)\cdot(\ln y-\ln x)\geq 0\ (\leq 0)$ on I^2 and according Lemma A2 we obtain the property of Schur-geometricaly convexity (Schur-geometricaly convexity) of K on I^2.

(v) The claim (i) implies that $\left(y^2\frac{\partial K(x,y)}{\partial y}-x^2\frac{x\partial K(x,y)}{\partial x}\right)(y-x)\geq 0\ (\leq 0)$ on I^2 and according Lemma A3 we obtain the property of Schur-harmonic convexity (Schur-harmonic concavity) of K on I^2.

(vi) Applying (8) for similarly (opposit) ordered functions f and g we have $\frac{\partial^2 K(x,y)}{\partial x\partial y}\leq 0\ (\geq 0)$. Lemma A4 implies that $K(x,y)$ is an $(1,1)$-concave (convex) function. □

References

[1] G.D.Anderson, M.K.Vamanamurthy, M.Vuorinen, Generalized convexity and inequalities,*Journal of Mathematical Analysis and Applications* , 335(2007), no.2, 1294 - 1308.

[2] Y. Chu and Y.-P. Lv, The Schur harmonic convexity of the Hamy symmetric function and its applications, *Journal of Inequalities and Applications*, Volume 2009, Article ID 838529, 10 pages.

[3] Y. Chu, X. Zhang and G. Wang, The Schur Geometrical Convexity of the Extended Mean Values, *Journal of Convex Analysis* 15 (2008), 707 - 718.

[4] V. Čuljak and J. Pečarić, Schur-convexity of Čebišev functional, *Mathematical Inequalities & Applications*, accepted for publication.

[5] V. Čuljak, Schur-convexity of the weighted Čebišev functional, *Journal of Mathematical Inequalities*, Volume **5**, Number 2 (2011), 213 - 217.

[6] H.-N. Shi and J. Zhang, Schur-convexity and Schur-geometric convexity of Čebišev functional, *RGMIA*, **30** no.3 (2000), 853 - 856.

[7] G.H. Hardy, J.E. Littlewood and G.Polya, Inequalities, *Cambrige*, 1934, 1952.

[8] A.W.Marshall and I. Olkin, Inequalities: Theory of Majorization and Its Applications, *Academic Press, Inc*, New York, 1979.

[9] D.S. Mitrinović, J.E. Pečarić and A.M. Fink, Classical and New Inequalities in Analysis, *Kluwer Academic Publishers*, 1993.

[10] C. P. Niculescu and L.-E. Persson, Convex Functions and Their Applications. A Contemporary Approach, *CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 23, Springer,* New York, NY, USA, 2006

[11] J. E. Pečarić, F. Proschan, and Y. L.Tong, Convex functions, partial orderings, and statistical applications, *Academic Press, Inc*, 1992.

[12] T. Popoviciu, Les functions convexes, *Herman and Cie Editeurs,* Paris, 1944.

[13] H.-N. Shi, Y.-M. Jiang and W.-D. Jiang, Schur-convexity and Schur-geometrically concavity of Gini means, *Computers & Mathematics with Applications* 57, 2 (2009), 266 - 274.

[14] W. Xia and Y. Chu, The Schur Harmonic Convexity of Lehmer Means, *International Mathematical Forum*, 4, 2009, no. 41, 2009 - 2015.

[15] X. M. Zhang, Geometrically Convex Functions, *Hefei: An'hui University Press*, 2004. (Chinese)

[16] X. M. Zhang, S-Geometric Convexity of a function involving MacLaurin's elementay symmetric mean, *Journal of inequalities in pure and applied mathematics,* vol. 8, iss. 2, art. 51, 2007

[17] L.-C. Wang, On the Monotonicity of Difference Generated y the Inequalities of Čebyšev Type, *Journal of Sichuan University, Natural Science Edition,* . **39(3)** (2002), 398-403 (Chinese).

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