

Polyominoes with nearly convex columns: A semi-directed model

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Abstract

Column-convex polyominoes are by now a well-explored model. So far, however, no attention has been given to polyominoes whose columns can have either one or two connected components. This little known kind of polyominoes seems not to be manageable as a whole. To obtain solvable models, one needs to introduce some restrictions. This paper is focused on polyominoes with hexagonal cells. The restrictions just mentioned are semi-directedness and an upper bound (say m) on the size of the gap within a column. As the upper bound m grows, the solution of the model tends to break into more and more cases. We computed the area generating functions for $m = 1$, $m = 2$ and $m = 3$. In this paper, the $m = 1$ and $m = 2$ models are solved in full detail. To keep the size of the paper within reasonable limits, the result for the $m = 3$ model is stated without proof. The $m = 1$, $m = 2$ and $m = 3$ models have rational area generating functions, as column-convex polyominoes do. (It is practically sure, although we leave it unproved, that the area generating functions are also rational for $m = 4$, $m = 5, \dots$) However, the growth constants of the new models are 4.114908 and more, whereas the growth constant of column-convex polyominoes is 3.863131.

Keywords: Polyomino, hexagonal-celled, nearly convex column, semi-directed, area generating function.

Math. Subj. Class.: 05B50, 05A15

1 Introduction

The enumeration of polyominoes is a topic of great interest to chemists, physicists and combinatorialists alike. In chemical terms, any polyomino (with hexagonal cells) is a possible benzenoid hydrocarbon. In physics, determining the number of n -celled polyominoes

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is related to the study of two-dimensional percolation phenomena. In combinatorics, polyominoes are of interest in their own right because several polyomino models have good-looking exact solutions.

Let a_n be the number of n -celled polyominoes. An exact formula for a_n seems unlikely to ever be found. However, there exist notable results on the quantity $c = \lim_{n \rightarrow \infty} \sqrt[n]{a_n}$. (This quantity is called the *growth constant*.) In the case of polyominoes with hexagonal cells, Vöge and Guttmann [21] gave a rigorous proof that the growth constant c exists and satisfies the inequality $4.8049 \leq c \leq 5.9047$. In the same paper, it is estimated that $c = 5.1831478(17)$.

In this paper, we are not going to improve the above-stated bounds on c . Instead, we are going to revisit *column-convex polyominoes*. The first to study this now-familiar model was Temperley [20] in 1956. Whether with square cells or with hexagonal cells, column-convex polyominoes have a rational area generating function. When cells are hexagons, the growth constant of column-convex polyominoes is 3.863131 (Klarner [17]). This growth constant remained a record (among polyomino models having *reasonably simple* exact solutions) until 1982, when Dhar, Phani and Barma [5] invented *directed site animals*. Directed site animals with step-set $A = \{(1, 0), (0, 1)\}$ can be viewed as a family of polyominoes with square cells, while directed site animals with step-set $B = \{(1, 0), (0, 1), (1, 1)\}$ can be viewed as a family of polyominoes with hexagonal cells. If the step-set is B , the growth constant of directed site animals is exactly 4. Incidentally, with either of the step-sets A and B , the area generating function of directed site animals is algebraic (which is rather surprising) and satisfies a simple quadratic equation. Later on, in 2002, Bousquet-Mélou and Rechnitzer [4] introduced *stacked directed animals* and *multi-directed animals*. (By now, each of those two models has been defined in two equivalent ways. There exist the original definitions [4], which use *heaps of pieces*, but there also exist recent definitions [10], which do not use heaps of pieces.) Stacked directed animals and multi-directed animals substantially generalize directed site animals. Whether the cells are squares or hexagons, the area generating function of stacked directed animals is degree-two algebraic, and the area generating function of multi-directed animals is not D-finite. When cells are hexagons, the growth constant of stacked directed animals is exactly 4.5, and the growth constant of multi-directed animals is 4.587894.

Although descended from directed animals, multi-directed animals are also a superset of column-convex polyominoes. (To be precise, this holds when cells are hexagons. It is not quite clear whether multi-directed animals with square cells are a superset of column-convex polyominoes with square cells.) Before the research presented here, multi-directed animals were unique in this respect. There existed pretty many polyomino models other than multi-directed animals, but, to my knowledge, not any of those other models is a superset of column-convex polyominoes. Indeed, stacked directed animals are not a superset of column-convex polyominoes. James, Jensen and Guttmann's *m-convex polygons* [16] are a superset of convex polyominoes, but are not a superset of column-convex polyominoes. Préa's *prudent polygons* [18, 6] are not even a superset of convex polyominoes. (The growth constant of prudent polygons is just 2 [1], but prudent polygons are nevertheless very popular in these times [3, 15, 19].)

The aim of this paper is to define a model which (a) generalizes column-convex polyominoes, (b) possesses a reasonably simple area generating function, and (c) possesses a high growth constant. In view of the facts stated above, we shall have to compete with just one pre-existing model, namely with multi-directed animals.

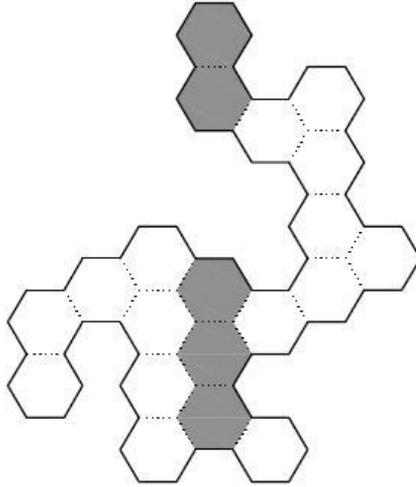


Figure 1: A hexagonal-celled polyomino.

In this paper, we actually introduce a sequence of generalizations of *hexagonal-celled column-convex polyominoes*. Namely, we define *level m cheesy polyominoes* ($m = 1, 2, 3, \dots$). At every level, our new model has a rational area generating function. (This is practically sure, although we do not give a proof.) However, as level increases, those rational generating functions quickly gain in size. In computing generating functions, we go up to level 3. The computations are done by using Bousquet-Mélou’s [2] and Svrtan’s [13] “turbo” version of the Temperley method [20]. The growth constants of level 1, level 2 and level 3 cheesy polyominoes turn out to be 4.114908, 4.231836 and 4.288631, respectively.

Thus, the growth constants of cheesy polyominoes are not as high as 4.587894, the growth constant of multi-directed animals. However, especially at level one, cheesy polyominoes have a simple area generating function, which cannot be said about multi-directed animals. Another good point of cheesy polyominoes is that their definition is short and immediate.

2 Definitions and conventions

There are three regular tilings of the Euclidean plane, namely the triangular tiling, the square tiling, and the hexagonal tiling. We adopt the convention that every square or hexagonal tile has two horizontal edges. In a regular tiling, a tile is often referred to as a *cell*. A plane figure P is a *polyomino* if P is a union of finitely many cells and the interior of P is connected. See Figure 1. Observe that, if a union of *hexagonal* cells is connected, then it possesses a connected interior as well.

Let P and Q be two polyominoes. We consider P and Q to be equal if and only if there exists a translation f such that $f(P) = Q$.

From now on, we concentrate on the hexagonal tiling. When we write “a polyomino”, we actually mean “a hexagonal-celled polyomino”.

Given a polyomino P , it is useful to partition the cells of P according to their horizontal

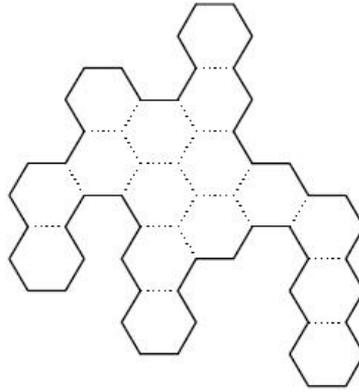


Figure 2: A column-convex polyomino.

projection. Each block of that partition is a *column* of P . Note that a column of a polyomino is not necessarily a connected set. An example of this is the highlighted column in Figure 1. On the other hand, it may happen that every column of a polyomino P is a connected set. In this case, the polyomino P is a *column-convex polyomino*. See Figure 2.

Let a and b be two adjacent columns of a polyomino P . Let the column b be the right neighbour of the column a . It is certain that column b has at least one edge in common with column a ; otherwise P could not be a connected set. However, there is no guarantee that every connected component of b has at least one edge in common with a . For example, in Figure 1, the upper component of the highlighted column has no edge in common with the previous column.

Suppose that P is such a polyomino that the first (*i.e.*, leftmost) column of P has no gap and that, in every pair of adjacent columns of P , every connected component of the right column has at least one edge in common with the left column. Then we say that P is a *rightward-semi-directed polyomino*.

A polyomino P is a *level m cheesy polyomino* if the following holds:

- P is a rightward-semi-directed polyomino,
- every column of P has at most two connected components,
- if a column of P has two connected components, then the gap between the components consists of at most m cells.

See Figure 3.

Level one cheesy polyominoes are a subset of level two cheesy polyominoes, level two cheesy polyominoes are a subset of level three cheesy polyominoes, and so on. As m tends to infinity, the set of level m cheesy polyominoes tends (in a certain sense) to the set of all polyominoes which are rightward-semi-directed and are made up of columns with at most two connected components.

The name “cheesy polyominoes” is intended to suggest that these polyominoes can have internal holes. At this point, it might be objected that there exists another model, called *directed animals*, in which holes (internal and not) occur more freely than in cheesy

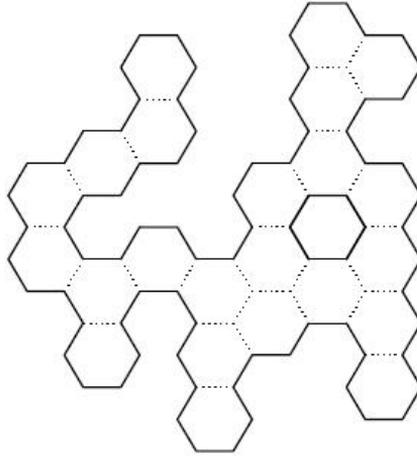
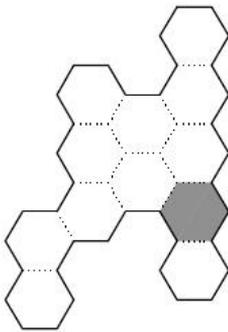


Figure 3: A level one cheesy polyomino.



The gray cell
is the pivot cell.

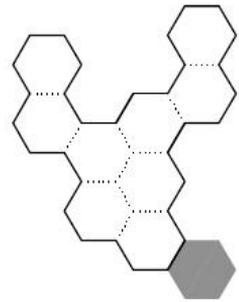


Figure 4: The pivot cell.

polyominoes. A column of a directed animal can have any number of holes, and the sizes of those holes are not subject to any limitations. However, the name of our model has 12 years of tradition¹. So, our model will retain the name “cheesy polyominoes” despite the fact that directed animals are arguably “cheesier”.

Let P be a polyomino with two or more columns. Then we define the *pivot cell* of P to be the lower right neighbour of the lowest cell of the second last column of P . See Figure 4. Observe that the pivot cell of a polyomino P is not necessarily contained in P . (Figure 4 shows the pivot cells of two column-convex polyominoes. However, the pivot cell is also defined for polyominoes which are not column-convex and have two or more columns.)

If a polyomino P is made up of n cells, we say that the *area* of P is n .

¹The contents of this paper exist since 1999. In that year, I defined cheesy polyominoes, explored them, and presented them at Mathematical Colloquium in Osijek [7], as well as at MATH/CHEM/COMP Course & Conference in Dubrovnik [8].

Let a be a column of a polyomino P . By the *height* of a we mean the number of those cells which make up a plus the number of those (zero or more) cells which make up the gaps of a . For example, in Figure 1, the highlighted column has height 7, and the next column to the left has height 4.

Let R be a set of polyominoes. By the *area generating function* of R we mean the formal sum

$$\sum_{P \in R} q^{\text{area of } P}.$$

By the *area and last column generating function* of R we mean the formal sum

$$\sum_{P \in R} q^{\text{area of } P} \cdot t^{\text{the height of the last column of } P}.$$

3 Column-convex polyominoes

The area generating function for column-convex polyominoes is known since 1967 (Klarner [17]). We shall state that well-known result without proof. The reader need not worry about this unexplained gap in our presentation. Namely, once the reader gets through the proof for level one cheesy polyominoes (given in the next section), it will be clear how the aforementioned gap can be bridged.

Let $A_1 = A_1(q)$ be the area generating function for column-convex polyominoes.

Proposition 3.1. *The area generating function for column-convex polyominoes is given by*

$$A_1 = \frac{q \cdot (1 - q)^3}{1 - 6q + 10q^2 - 7q^3 + q^4}.$$

Corollary 3.2. *The number of n -celled column-convex polyominoes $[q^n]A_1$ has the asymptotic behaviour*

$$[q^n]A_1 \sim 0.188420 \cdot 3.863131^n.$$

Thus, the growth constant of column-convex polyominoes is 3.863131.

4 Level one cheesy polyominoes

Counting level one cheesy polyominoes by area is similar to counting column-convex polyominoes by area.

Let $C = C(q, t)$ be the area and last column generating function for level one cheesy polyominoes. Let $C_1 = C(q, 1)$ and $D_1 = \frac{\partial C}{\partial t}(q, 1)$.

Let T be the set of all level one cheesy polyominoes. We write T_α for the set of level one cheesy polyominoes which have only one column. Let

$$\begin{aligned} T_\beta &= \{P \in T \setminus T_\alpha : \text{the last column of } P \text{ has no hole,} \\ &\quad \text{and the pivot cell of } P \text{ is contained in } P\}, \\ T_\gamma &= \{P \in T \setminus T_\alpha : \text{the last column of } P \text{ has no hole,} \\ &\quad \text{and the pivot cell of } P \text{ is not contained in } P\}, \text{ and} \\ T_\delta &= \{P \in T \setminus T_\alpha : \text{the last column of } P \text{ has a hole}\}. \end{aligned}$$

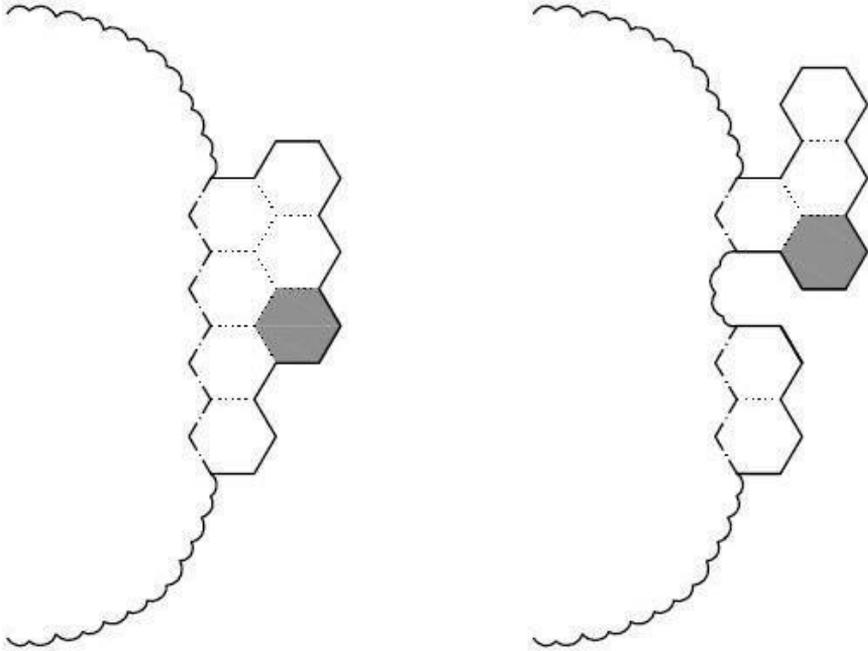


Figure 5: The last two columns of two elements of T_γ .

The sets T_α , T_β , T_γ and T_δ form a partition of T . We write C_α , C_β , C_γ and C_δ to denote the parts of the series C that come from the sets T_α , T_β , T_γ and T_δ , respectively.

We have

$$C_\alpha = qt + (qt)^2 + (qt)^3 + \dots = \frac{qt}{1 - qt} \tag{4.1}$$

and

$$C_\beta = \frac{qt}{(1 - qt)^2} \cdot C_1. \tag{4.2}$$

Consider the following situation. A cheesy polyomino P ends with a column I . We are creating a new column to the right of I , and the result should be an element of T_γ . Then, whether or not the column I has a hole, we can put the lowest cell of the new column in exactly m places, where m is the height of I . See Figure 5. Hence

$$C_\gamma = \frac{qt}{1 - qt} \cdot D_1. \tag{4.3}$$

Let us move on to another situation. A cheesy polyomino P ends with a column J . We are creating a new column to the right of J , and the result should be an element of T_δ . Then, whether or not the column J has a hole, we can put the hole of the new column in exactly $n - 1$ places, where n is the height of J . See Figure 6. The new column is

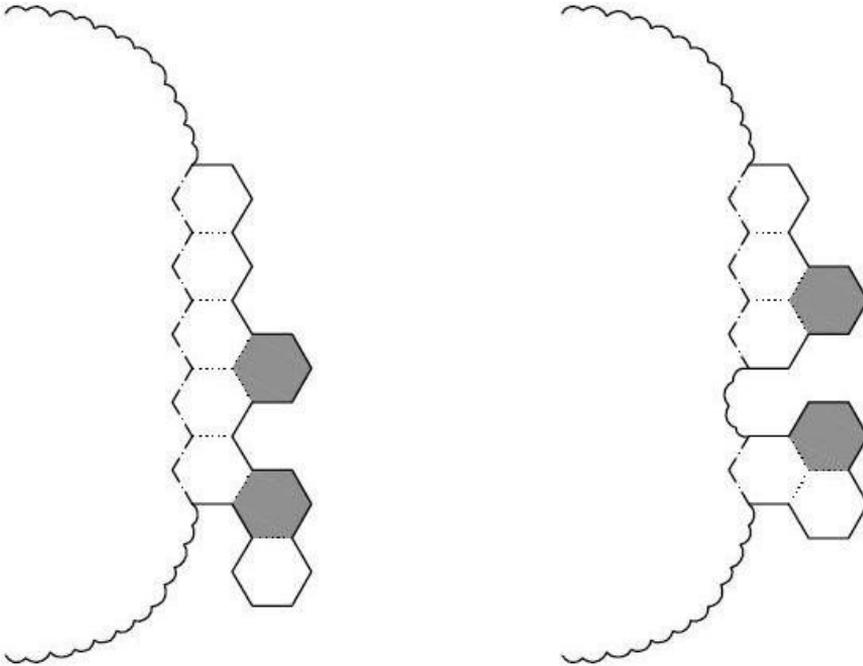


Figure 6: The last two columns of two elements of T_δ .

made up of $i \in \{1, 2, 3, \dots\}$ cells lying below the hole, of a hole of height one, and of $j \in \{1, 2, 3, \dots\}$ cells lying above the hole. Altogether,

$$C_\delta = \frac{qt}{1-qt} \cdot t \cdot \frac{qt}{1-qt} \cdot (D_1 - C_1) = \frac{q^2 t^3}{(1-qt)^2} \cdot (D_1 - C_1). \tag{4.4}$$

Since $C = C_\alpha + C_\beta + C_\gamma + C_\delta$, Eqs. (4.1), (4.2), (4.3) and (4.4) imply that

$$C = \frac{qt}{1-qt} + \frac{qt}{(1-qt)^2} \cdot C_1 + \frac{qt}{1-qt} \cdot D_1 + \frac{q^2 t^3}{(1-qt)^2} \cdot (D_1 - C_1). \tag{4.5}$$

Setting $t = 1$, from Eq. (4.5) we obtain

$$C_1 = \frac{q}{1-q} + \frac{q}{(1-q)^2} \cdot C_1 + \frac{q}{1-q} \cdot D_1 + \frac{q^2}{(1-q)^2} \cdot (D_1 - C_1).$$

Differentiating Eq. (4.5) with respect to t and then setting $t = 1$, we obtain

$$\begin{aligned} D_1 = & \frac{q}{1-q} + \frac{q^2}{(1-q)^2} + \left[\frac{q}{(1-q)^2} + \frac{2q^2}{(1-q)^3} \right] \cdot C_1 \\ & + \left[\frac{q}{1-q} + \frac{q^2}{(1-q)^2} \right] \cdot D_1 + \left[\frac{3q^2}{(1-q)^2} + \frac{2q^3}{(1-q)^3} \right] \cdot (D_1 - C_1). \end{aligned}$$

Thus, we have a system of two linear equations in two unknowns, C_1 and D_1 . By solving the system, we get this proposition.

Proposition 4.1. *The area generating function for level one cheesy polyominoes is given by*

$$C_1 = \frac{q(1 - 3q + q^2)}{1 - 6q + 8q^2 - q^3}.$$

Of course, the denominator of C_1 has three complex roots. Those roots are² $r_1 = 0.243019$, $r_2 = 0.572771$ and $r_3 = 7.184210$. The root with smallest absolute value is $r_1 = 0.243019$, and $\frac{1}{r_1}$ is equal to 4.114908. Therefore, the coefficient of q^n in C_1 (denoted $[q^n]C_1$) has the asymptotic behaviour $[q^n]C_1 \sim k \cdot 4.114908^n$, where k is a constant. To find the value of k , we decompose C_1 into partial fractions. The partial fraction involving r_1 turns out to be $-\frac{0.035038}{q-0.243019}$. In the Taylor series expansion of $-\frac{0.035038}{q-0.243019}$, the coefficient of q^n is equal to $0.144176 \cdot 4.114908^n$. Thus, $k = 0.144176$. We have got the following result.

Corollary 4.2. *The number of n -celled level one cheesy polyominoes $[q^n]C_1$ has the asymptotic behaviour*

$$[q^n]C_1 \sim 0.144176 \cdot 4.114908^n.$$

Thus, the growth constant of level one cheesy polyominoes is 4.114908. We observe a considerable increase with respect to 3.863131, the growth constant of column-convex polyominoes.

5 Level two cheesy polyominoes

In this enumeration, if the last column of a polyomino has two connected components, we sometimes need to record not only the overall height of the last column, but also the height of the last column's upper component and the height of the last column's lower component. Hence, in addition to the "old" variables q and t , we introduce two new variables, u and v . As before, the exponent of q is the area and the exponent of t is the overall height of the last column³. The exponent of u is the height of the upper component of the last column, and the exponent of v is the height of the lower component of the last column.

The two main generating functions in this enumeration are $E = E(q, t)$ and $G = G(q, t, u, v)$. Those generating functions are used for the following purposes:

- E is a generating function for level two cheesy polyominoes whose last column either has no hole or has a one-celled hole,
- G is a generating function for level two cheesy polyominoes whose last column has a two-celled hole.

Let $E_1 = E(q, 1)$, $F_0 = \frac{\partial E}{\partial t}(q, 0)$, $F_1 = \frac{\partial E}{\partial t}(q, 1)$, $G_1 = G(q, 1, 1, 1)$, $H_1 = \frac{\partial G}{\partial t}(q, 1, 1, 1)$, $I_0 = \frac{\partial G}{\partial u}(q, 1, 0, 1)$, and $J_0 = \frac{\partial G}{\partial v}(q, 1, 1, 0)$.

²Those roots have infinitely many digits. Since here we do not have an infinite amount of space, we shall round the roots to six decimal places.

³Recall what do we mean by the height of a column: in Figure 1, the highlighted column has height 7, and the next column to the left has height 4.

Let U be the set of those level two cheesy polyominoes whose last column either has no hole or has a one-celled hole. Let V be the set of those level two cheesy polyominoes whose last column has a two-celled hole. For $P \in U \cup V$, we define the *body* of P to be all of P , except the rightmost column of P .

We are now going to partition the set U into seven subsets and the set V into two subsets. Let

- $U_\alpha = \{P \in U : P \text{ has only one column}\},$
- $U_\beta = \{P \in U \setminus U_\alpha : \text{the body of } P \text{ lies in } U, \text{ the last column of } P \text{ has no hole, and the pivot cell of } P \text{ is contained in } P\},$
- $U_\gamma = \{P \in U \setminus U_\alpha : \text{the body of } P \text{ lies in } U, \text{ the last column of } P \text{ has no hole, and the pivot cell of } P \text{ is not contained in } P\},$
- $U_\delta = \{P \in U \setminus U_\alpha : \text{the body of } P \text{ lies in } U, \text{ and the last column of } P \text{ has a hole}\},$
- $U_\epsilon = \{P \in U \setminus U_\alpha : \text{the body of } P \text{ lies in } V, \text{ the last column of } P \text{ has no hole, and the pivot cell of } P \text{ is contained in } P\},$
- $U_\zeta = \{P \in U \setminus U_\alpha : \text{the body of } P \text{ lies in } V, \text{ the last column of } P \text{ has no hole, and the pivot cell of } P \text{ is not contained in } P\},$ and
- $U_\eta = \{P \in U \setminus U_\alpha : \text{the body of } P \text{ lies in } V, \text{ and the last column of } P \text{ has a hole}\}.$

Let

- $V_\alpha = \{P \in V : \text{the body of } P \text{ lies in } U\},$ and
- $V_\beta = \{P \in V : \text{the body of } P \text{ lies in } V\}.$

It is clear that the sets $U_\alpha, U_\beta, \dots, U_\eta$ form a partition of U , and that the sets V_α and V_β form a partition of V . We shall write $E_\alpha, E_\beta, \dots, E_\eta$ for the parts of the series E that come from the sets $U_\alpha, U_\beta, \dots, U_\eta$, respectively. Also, we shall write G_α and G_β for the parts of the series G that come from the sets V_α and V_β , respectively.

As in Section 4, we have

$$E_\alpha = \frac{qt}{1 - qt}, \tag{5.1}$$

$$E_\beta = \frac{qt}{(1 - qt)^2} \cdot E_1, \tag{5.2}$$

$$E_\gamma = \frac{qt}{1 - qt} \cdot F_1, \tag{5.3}$$

$$E_\delta = \frac{q^2t^3}{(1 - qt)^2} \cdot (F_1 - E_1), \tag{5.4}$$

$$E_\epsilon = \frac{qt}{(1 - qt)^2} \cdot G_1. \tag{5.5}$$

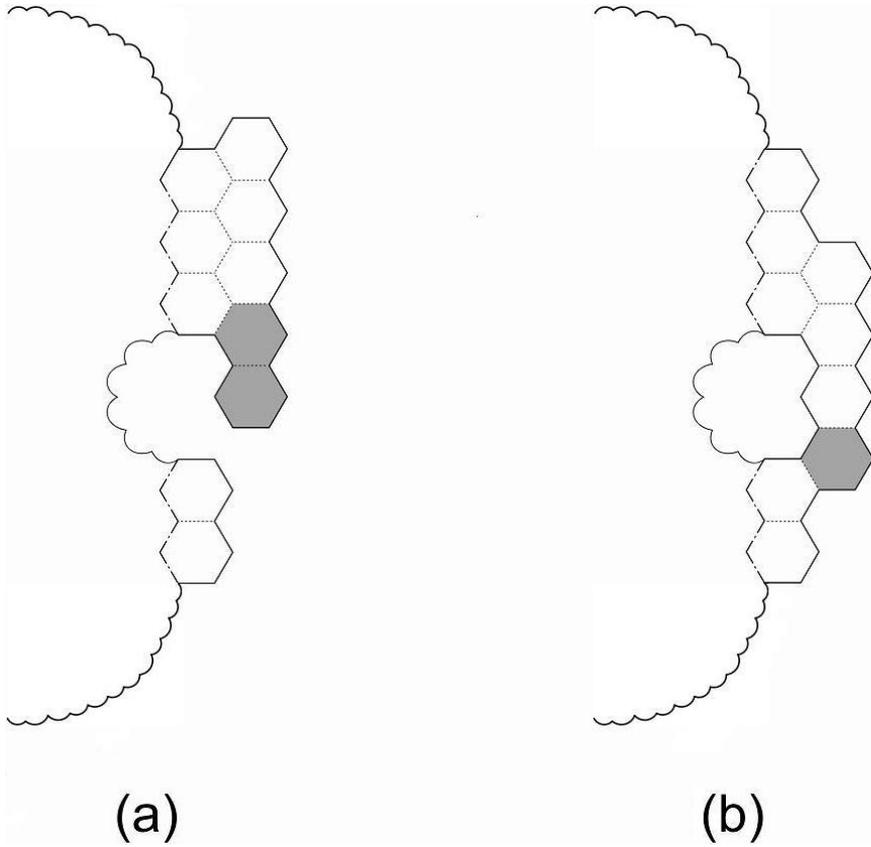


Figure 7: (a) For this choice of the lowest cell, the new column must be at least two cells high. (b) For this choice of the lowest cell, the new column does not have to be at least two cells high.

The functional equation for E_ζ is more interesting. Let P be an element of V , let I be the last column of P , and let m be the height of I . Suppose that we are creating a new column to the right of I , and that the result should be an element of U_ζ . To the right of I there is one cell (say c) which shares an edge with each of the two cells forming the hole of I . If we choose c as the lowest cell of the new column, then the new column will have to be at least two cells high. Otherwise $P \cup$ (the new column) will not be a polyomino. See Figure 7. In addition to c , there are $m - 1$ other choices for the lowest cell of the new column. For each of these $m - 1$ choices, $P \cup$ (the new column) is a polyomino regardless of how many cells the new column has. Altogether, we have

$$E_\zeta = \frac{q^2 t^2}{1 - qt} \cdot G_1 + \frac{qt}{1 - qt} \cdot (H_1 - G_1). \tag{5.6}$$

We proceed to E_η . Let P be an element of V , let J be the last column of P , and let n be the height of J . Suppose that we are creating a new column to the right of J , and

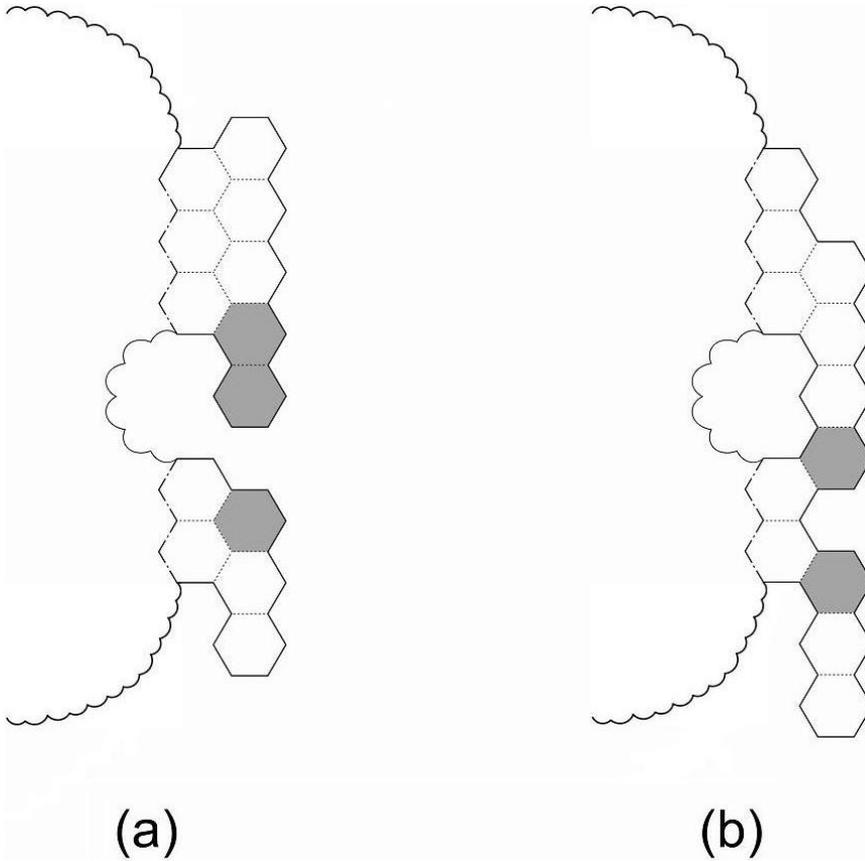


Figure 8: (a) If the one-celled hole is in this position, the upper component of the new column must have at least two cells. (b) If the one-celled hole is in this position, the upper and lower components of the new column can be of any sizes.

that the result should be an element of U_η . One of the choices for the hole of the new column is the upper right neighbour of the top cell of the lower component of J . For this choice, the upper component of the new column has to have at least two cells. Otherwise $P \cup$ (the new column) is not a polyomino. See Figure 8. We can also choose the hole of the new column as the lower right neighbour of the bottom cell of the upper component of J . Then, in order for $P \cup$ (the new column) to be a polyomino, the lower component of the new column has to have at least two cells. In addition to the two ways just considered, there exist $n - 3$ other ways to choose the hole of the new column. For each of those $n - 3$ ways, $P \cup$ (the new column) is a polyomino regardless of the sizes of the upper and lower components of the new column. Thus, we have

$$E_\eta = \frac{2q^3t^4}{(1-qt)^2} \cdot G_1 + \frac{q^2t^3}{(1-qt)^2} \cdot (H_1 - 3G_1). \tag{5.7}$$

Of course, the series $E_\alpha, E_\beta, \dots, E_\eta$ sum up to E . Therefore, the summation of Eqs.

(5.1)–(5.7) gives

$$\begin{aligned}
 E &= \frac{qt}{1-qt} + \frac{qt}{(1-qt)^2} \cdot E_1 + \frac{qt}{1-qt} \cdot F_1 + \frac{q^2t^3}{(1-qt)^2} \cdot (F_1 - E_1) \\
 &+ \frac{qt}{(1-qt)^2} \cdot G_1 + \frac{q^2t^2}{1-qt} \cdot G_1 + \frac{qt}{1-qt} \cdot (H_1 - G_1) \\
 &+ \frac{2q^3t^4}{(1-qt)^2} \cdot G_1 + \frac{q^2t^3}{(1-qt)^2} \cdot (H_1 - 3G_1). \tag{5.8}
 \end{aligned}$$

We also need to establish a functional equation for G . Let P be an element of U and let c be a column with a two-celled hole. Suppose that we want to glue c to P so that $P \cup c$ lies in V_α , and so that P and c are the body and the last column of $P \cup c$, respectively. In how many ways P and c can be glued together? In principle, the number of ways is (the height of the last column of P) minus two. See Figure 9. However, if P ends with a one-celled column, then we can glue c to P in zero ways, and not in minus one ways. Thus, we have

$$G_\alpha = \frac{q^2t^4uv}{(1-qtu)(1-qtv)} \cdot (F_1 - 2E_1 + F_0). \tag{5.9}$$

In the case of G_β , it is convenient to use overcounting. That is, we are going to “mistakenly” count too much, and then subtract the parts which do not belong. Let P be an element of V and let c be a column with a two-celled hole. Suppose that we want to glue c to P so that $P \cup c$ lies in V_β , and so that P and c are the body and the last column of $P \cup c$, respectively. In how many ways P and c can be glued together? First, there are (the height of the last column of P) minus two ways to satisfy these two necessary conditions:

- the bottom cell of the upper component of c is either identical with or lies lower than the upper right neighbour of the top cell of the last column of P , and
- the top cell of the lower component of c is either identical with or lies higher than the lower right neighbour of the bottom cell of the last column of P .

See Figure 10.

So, if there were no special cases, then G_β would be equal to

$$\frac{q^2t^4uv}{(1-qtu)(1-qtv)} \cdot (H_1 - 2G_1). \tag{5.10}$$

However, special cases do exist. There are two of them:

1. The upper component of the last column of $P \in V$ has at least two cells and the lower component of the two-component column c has just one cell.
2. The lower component of the last column of $P \in V$ has at least two cells and the upper component of the two-component column c has just one cell.

In case 1, it is (so to speak) dangerous to glue c to P in such a way that the one-celled lower component of c becomes a common neighbour of the two cells which form the hole of the last column of P . This dangerous operation produces an object which is not a polyomino and hence does not lie in V_β .

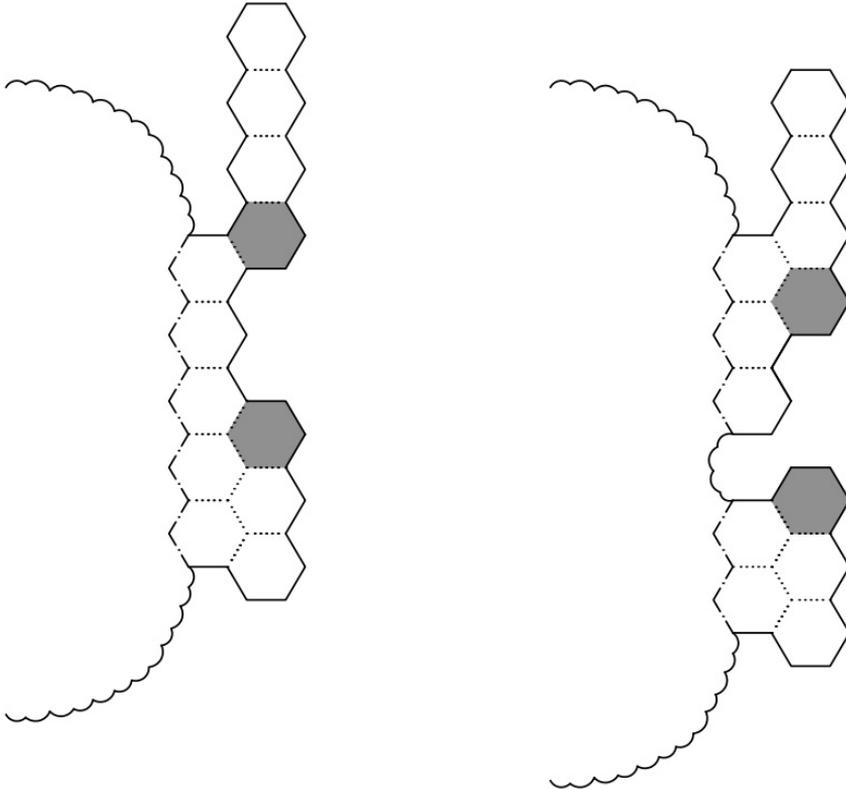


Figure 9: The last two columns of two elements of V_α .

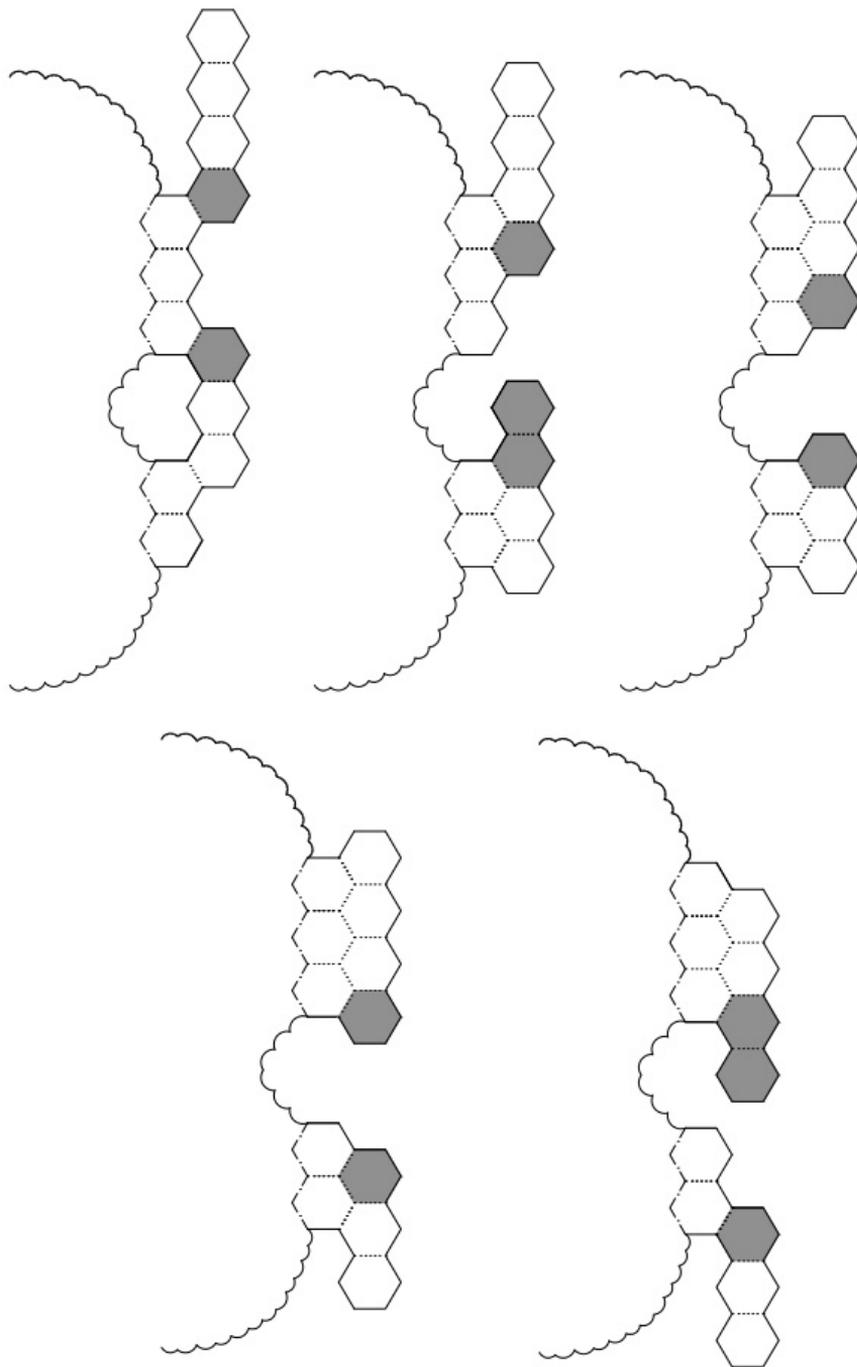


Figure 10: The last two columns of five elements of V_β .

In case 2, it is dangerous to glue c to P in such a way that the one-celled upper component of c becomes a common neighbour of the two cells which form the hole of the last column of P . Again, the dangerous operation produces an object which is not a polyomino and hence does not lie in V_β .

Now, Eq. (5.10) is actually a generating function for the union of V_β with the set of objects produced by the two dangerous operations. The generating function for the objects produced by the first dangerous operation is

$$\frac{q^2 t^4 uv}{1 - qtu} \cdot (G_1 - I_0). \tag{5.11}$$

The generating function for the objects produced by the second dangerous operation is

$$\frac{q^2 t^4 uv}{1 - qtv} \cdot (G_1 - J_0). \tag{5.12}$$

Subtracting Eqs. (5.11) and (5.12) from Eq. (5.10), we obtain

$$G_\beta = \frac{q^2 t^4 uv}{(1 - qtu)(1 - qtv)} \cdot (H_1 - 2G_1) - \frac{q^2 t^4 uv}{1 - qtu} \cdot (G_1 - I_0) - \frac{q^2 t^4 uv}{1 - qtv} \cdot (G_1 - J_0). \tag{5.13}$$

Since $G = G_\alpha + G_\beta$, Eqs. (5.9) and (5.13) imply that

$$G = \frac{q^2 t^4 uv}{(1 - qtu)(1 - qtv)} \cdot (F_1 - 2E_1 + F_0 + H_1 - 2G_1) - \frac{q^2 t^4 uv}{1 - qtu} \cdot (G_1 - I_0) - \frac{q^2 t^4 uv}{1 - qtv} \cdot (G_1 - J_0). \tag{5.14}$$

Using the computer algebra system *Maple*, from Eqs. (5.8) and (5.14) we obtained a system of seven linear equations in seven unknowns: $E_1, F_0, F_1, G_1, H_1, I_0$ and J_0 . Rather than write down these seven equations (some of which are a bit cumbersome), here below we give a list of recipes. Recipe no. k ($k = 1, 2, \dots, 7$) tells how to obtain the k th equation of the linear system.

1. In Eq. (5.8), set $t = 1$.
2. Differentiate Eq. (5.8) with respect to t and then set $t = 0$.
3. Differentiate Eq. (5.8) with respect to t and then set $t = 1$.
4. In Eq. (5.14), set $t = u = v = 1$.
5. Differentiate Eq. (5.14) with respect to t and then set $t = u = v = 1$.
6. Differentiate Eq. (5.14) with respect to u . Then set $t = v = 1$ and $u = 0$.
7. Differentiate Eq. (5.14) with respect to v . Then set $t = u = 1$ and $v = 0$.

The computer algebra quickly solved the linear system and then summed the generating functions E_1 and G_1 . The result can be seen in the following proposition.

Proposition 5.1. *The area generating function for level two cheesy polyominoes is given by*

$$K = \frac{q \cdot (1 - 6q + 11q^2 - 6q^3 - q^4 - 3q^6 + 5q^7 + 4q^8 - 3q^9 - 3q^{10})}{1 - 9q + 27q^2 - 31q^3 + 8q^4 + 4q^5 - 2q^6 + 16q^7 - 5q^8 - 16q^9 - 2q^{10} + 5q^{11}}.$$

Corollary 5.2. *The number of n -celled level two cheesy polyominoes has the asymptotic behaviour*

$$[q^n]K \sim 0.121043 \cdot 4.231836^n.$$

Thus, the growth constant of level two cheesy polyominoes is 4.231836.

6 Level three cheesy polyominoes

The enumeration of level three cheesy polyominoes is similar to the enumeration of level two cheesy polyominoes. However, there are still more cases than before. In the enumeration of level two cheesy polyominoes, the total number of cases was 9 (because we partitioned the set U into 7 subsets and the set V into 2 subsets). At level three, the total number of cases is 16. We deem it reasonable to skip those 16 cases and only state the final result.

Proposition 6.1. *The area generating function for level three cheesy polyominoes is given by*

$$L = \frac{M}{N},$$

where

$$\begin{aligned} M = & q \cdot (1 - 11q + 49q^2 - 114q^3 + 146q^4 - 94q^5 + 5q^6 + 71q^7 - 143q^8 \\ & + 176q^9 - 154q^{10} + 100q^{11} + 24q^{12} - 121q^{13} + 90q^{14} - 61q^{15} + 19q^{16} \\ & + 58q^{17} - 32q^{18} - 31q^{19} + 37q^{20} + 14q^{21} - 43q^{22} - 4q^{23} + 21q^{24} \\ & - q^{25} - 5q^{26}) \end{aligned}$$

and

$$\begin{aligned} N = & 1 - 14q + 80q^2 - 243q^3 + 423q^4 - 413q^5 + 174q^6 + 106q^7 - 350q^8 \\ & + 533q^9 - 546q^{10} + 427q^{11} - 148q^{12} - 261q^{13} + 383q^{14} - 253q^{15} \\ & + 158q^{16} + 57q^{17} - 181q^{18} + 10q^{19} + 115q^{20} - 49q^{21} - 96q^{22} \\ & + 93q^{23} + 49q^{24} - 54q^{25} - 12q^{26} + 12q^{27} + q^{28}. \end{aligned}$$

Corollary 6.2. *The number of n -celled level three cheesy polyominoes has the asymptotic behaviour*

$$[q^n]L \sim 0.108270 \cdot 4.288631^n.$$

Area	Column-convex polyominoes	Level 1 cheesy polyominoes	Level 2 cheesy polyominoes	Level 3 cheesy polyominoes
1	1	1	1	1
2	3	3	3	3
3	11	11	11	11
4	42	43	43	43
5	162	173	174	174
6	626	705	718	719
7	2419	2889	2996	3012
8	9346	11867	12579	12727
9	36106	48795	52996	54067
10	139483	200723	223705	230464
11	538841	825845	945324	984477
12	2081612	3398081	3997185	4211222

Table 1: Here is how many polyominoes of a given type have 1, 2, ..., 12 cells.

7 Taylor expansions and the limit value of the growth constants

To see how many polyominoes of a given type have 1, 2, 3, ... cells, we expanded the area generating functions into Taylor series. The results are shown in Table 1.

Next: How do the growth constants behave when level tends to infinity? Our database is too small for making precise estimates. Anyway, we know that the growth constant of column-convex polyominoes is 3.863, while the growth constants of level one, level two and level three cheesy polyominoes are 4.115, 4.232 and 4.289, respectively. Computing the first differences, we get the numbers $4.115 - 3.863 = 0.252$, $4.232 - 4.115 = 0.117$, and $4.289 - 4.232 = 0.057$. Now, the sequence 0.252, 0.117, 0.057 is a little similar to a geometric sequence with common ratio $\frac{1}{2}$. Hence, the limit value of the growth constants of cheesy polyominoes might be about $4.289 + 0.057 = 4.346$.

8 Conclusion

This paper is concerned with polyominoes on the hexagonal lattice. For every $m \in \mathbb{N}$, we have defined a set of polyominoes called level m cheesy polyominoes. A polyomino P is a level m cheesy polyomino if the following holds:

1. P is a rightward-semi-directed polyomino,
2. every column of P has at most two connected components,
3. if a column of P has two connected components, then the gap between the components consists of at most m cells.

Column-convex polyominoes are a subset of level one cheesy polyominoes and, for every $m \in \mathbb{N}$, level m cheesy polyominoes are a subset of level $m + 1$ cheesy polyominoes.

For every $m \in \mathbb{N}$, level m cheesy polyominoes have a rational area generating function. We have computed the area generating functions for levels one, two and three. At

those three levels, the number of n -celled cheesy polyominoes is asymptotically equal to $0.1441 \dots \times 4.1149 \dots^n$, to $0.1210 \dots \times 4.2318 \dots^n$, and to $0.1082 \dots \times 4.2886 \dots^n$, respectively. For comparison, the number of n -celled column-convex polyominoes is asymptotically equal to $0.1884 \dots \times 3.8631 \dots^n$. The number of n -celled multi-directed animals behaves asymptotically as $constant \times 4.5878 \dots^n$ [4]. (At present, multi-directed animals are the largest exactly solved class of polyominoes. However, multi-directed animals are not a superset of level one cheesy polyominoes.) The number of all n -celled polyominoes behaves asymptotically as $\frac{0.2734 \dots}{n} \times 5.1831 \dots^n$ [21].

9 Fresh news

The present paper is being published after many adventures. Meanwhile, this author has generalized column-convex polyominoes in several other ways. The definition of cheesy polyominoes (stated in Section 8) consists of three requirements. Now, in [9], the requirement that polyominoes are rightward-semi-directed is somehow relaxed. After this relaxation, the area generating functions are still rational but are somewhat less simple than in the case of cheesy polyominoes. In [11, 12], the rightward-semi-directedness requirement is completely removed. That is, the definition of the model is reduced to requirements No. 2 and 3 of the definition of cheesy polyominoes. For the model so defined, the area generating function is not a rational function, but a complicated q -series. It is also possible not to require rightward-semi-directedness and, at the same time, allow two-component columns to have gaps of all sizes. The only remaining requirement is then the second one, “every column of P has at most two connected components”. The said requirement by itself defines an unsolvable model, but that model can be made solvable by introducing a new requirement. Indeed, in [11, 14], we forbid runs of two or more consecutive two-component columns, and the area generating function is again a complicated, but computable, q -series.

All this does not mean, however, that cheesy polyominoes are no longer interesting. In particular, it remains worthy of attention that level one cheesy polyominoes, which are significantly more numerous than column-convex polyominoes, have an area generating function which is even a bit simpler than the area generating function of column-convex polyominoes.

Let us conclude with a few words about possibilities for future work. As far as I can see, if requirement No. 2 is replaced by “every column of P has at most three connected components”, the resulting model is still solvable, but if requirement No. 2 is replaced by “every column of P has at most four connected components”, the resulting model is unsolvable.

I think that, already at level one, cheesy polyominoes cannot be enumerated by perimeter. Namely, the perimeter generating function has zero radius of convergence, as can be proved by adapting an argument given by Tony Guttmann in Section 9 of [11].

Cheesy polyominoes with square cells are an object which, in principle, can be studied. It ought to be mentioned, however, that hexagonal-celled cheesy polyominoes behave somewhat better than their square-celled counterpart. Solving the square-celled level one model requires almost as much effort as solving the hexagonal-celled level two model. The same holds for the square-celled versions of the models of [9, 12].

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