Polynomial WENO approximation with applications

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Abstract

Weighted essentially non-oscillatory (WENO) reconstruction procedure is used for a non-oscillatory approximation of a function.

- **WENO reconstruction procedure**: constructs a rational approximation of a function $v(x)$ based on known cell averages of function $v(x)$.
- **WENO interpolation procedure**: constructs a rational interpolating function $v(x)$ based on pointwise values of function $v(x)$.

Those approximations are essentially non-oscillatory and high order accurate for smooth enough function $v(x)$. For the case of reconstruction, rational function obtained by the classical WENO reconstruction contains poles. Therefore, if one needs to obtain a value of function $v(x)$ in the interior of the numerical cell, the standard WENO procedure needs special treatment to overcome instabilities that can occur.

We will present a **new polynomial version of the WENO procedure** that provides an elegant solution to those problems by constructing an approximating polynomial that is smooth, non-oscillatory, and high order accurate.

Polynomial WENO 1. step:

Based on average function values $\bar{v}_i$ form polynomials $p_i(x)$ such that: $p_i(x) = v(x) + \mathcal{O}(\Delta x^r)$, $x \in I_i$. There are r polynomials that satisfy:

$$\frac{1}{\Delta x}\int_{I_j} p_i^s(\xi)d\xi = \bar{v}_j, \; j = i-r+1+s, \dots, i+s.$$

Values of the primitive function $V(x) = \int_a^x v(\xi)d\xi$, are exactly known for cell edges.

Let $P_i^s(x)$ be an polynomial of degree r that interpolates points of $V(x)$ at cell edges $x_{i-r+s+\frac{1}{2}}, \dots, x_{i+s+\frac{1}{2}}$.

Polynomial WENO 2. step:

Form ideal weights (polynomial functions) $C_{r,s}(x)$, $s = 0, \dots, r-1$ to obtain higher degree polynomial $Q_i(x)$:

$$Q_i(x) = \sum_{s=0}^{r-1} C_{r,s}(x) P_i^s(x).$$

$Q_i(x)$ can be very accurate approximation of $v(x)$, but it will be highly oscillatory if $v(x)$ is not smooth enough. The oscillations of polynomials $P_i^s(x)$ can be measured by using smoothness indicators:

$$IS_{r,s} = \sum_{l=2}^{r-1} \int_{I_i} \Delta x^{2l-1} \left(\frac{d^l P_i^s(x)}{dx^l}\right)^2 dx.$$

Polynomial WENO 3. step:

The standard WENO approach is based on the idea of replacing linear weight $C_{r,s}(x)$ with nonlinear weighting factor $\Omega_{r,s}(x)$:

$$\Omega_{r,s} = \frac{A_{r,s}}{\sum_{j=0}^{r-1} A_{r,j}}, \; A_{r,s} = \frac{C_{r,s}}{(\epsilon + IS_{r,s})^2}.$$

In the **polynomial WENO approach** we suggest to replace the weighting factors with the interpolating polynomials in the following way: Subdivide the interval I_i into a finer grid $\tilde{X}_i = (x_{i,j})_{0\le j\le r-1}$ of evenly spaced nodes. Replace the weighting functions $\Omega_{r,s}(x)$ with:

$$\tilde{\Omega}_{r,s}(x) = \sum_{j=0}^{r-1} l_j^{\tilde{X}_i}(x) \cdot \Omega_{r,s}(x_{i,j}),$$

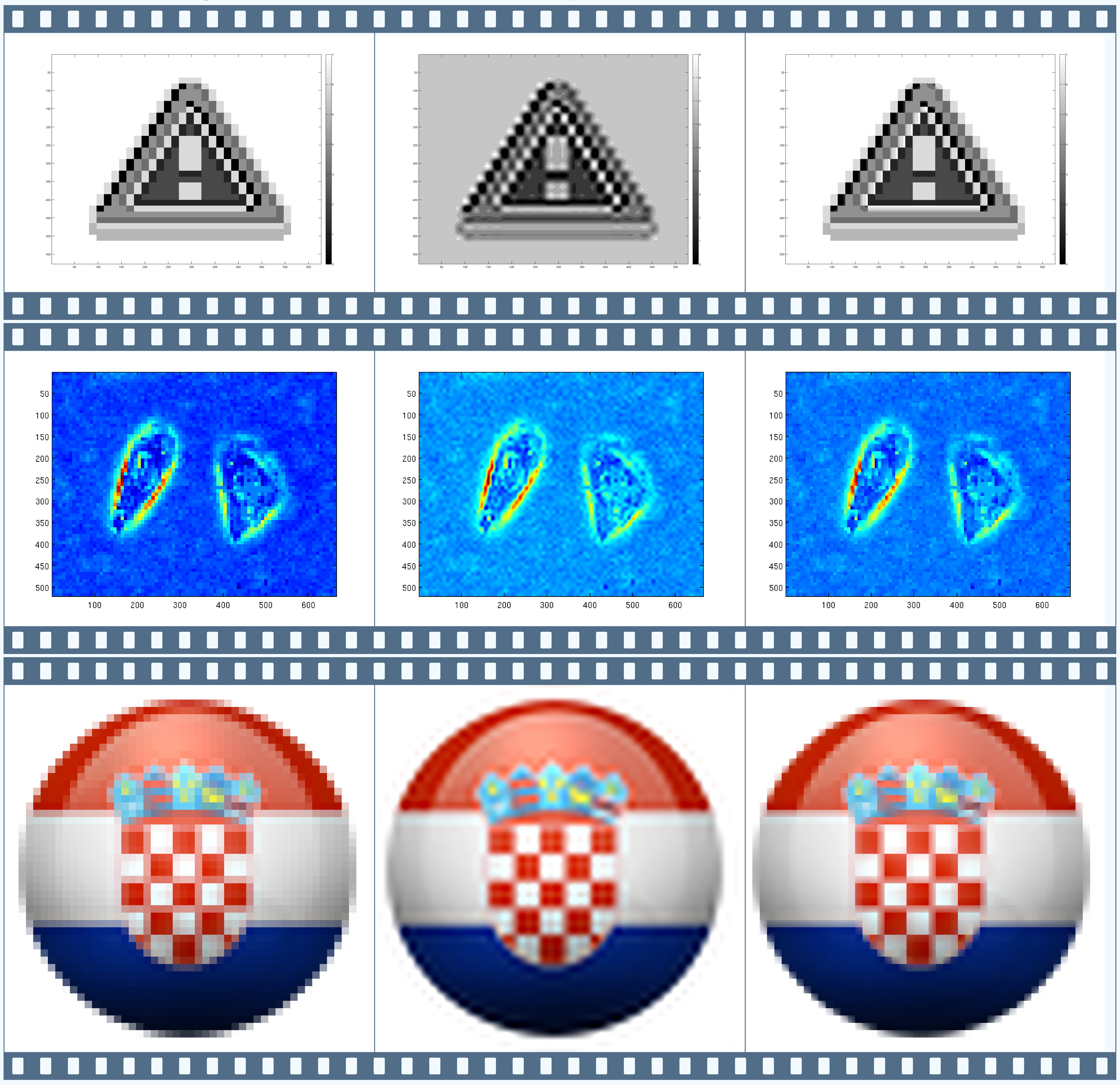
where $l_j^{\tilde{X}_i}(x)$ is j-th Lagrange cardinal polynomial associated to the grid $\tilde{X}_i$.

Problem

Increase the image resolution by splitting each pixel into subpixels. Calculate subpixel values using some 1D interpolation algorithm applied dimension by dimension.

Results

Original | Cubic Spline | Cubic WENO



When cubic spline interpolation is used, quite blurred figure arises. The contours are not clear due to the oscillations that appear when sharp changes in color between neighboring pixels occur.

On the other hand, when polynomial WENO interpolation is applied, sharp contours in figure are preserved. Even more, it seems that the brightening effect arises.

Polynomial WENO 4. step:

WENO polynomial **interpolation** for $V(x)$:

$$P_r(x) = \sum_{s=0}^{r-1} \tilde{\Omega}_{r,s}(x) \cdot P_i^s(x),$$

WENO polynomial **reconstruction** for $v(x)$:

$$p_r(x) = P_r'(x).$$

WENO cell **subdivision**:
The average function values satisfy:

$$\bar{v}_i = \frac{1}{\Delta x}\int_{I_i} P_r'(\xi)d\xi = \frac{1}{\Delta x}\left(P_r(x_{i+1/2}) - P_r(x_{i-1/2})\right).$$

Each cell I_i is subdivided in to n subcells $I_{i,j}$ and the corresponding average values are evaluated as:

$$\bar{v}_{i,j} = \frac{1}{n\,\Delta x}\int_{I_{i,j}} P_r'(\xi)d\xi, \; j = 1\dots n.$$

Image reconstruction:

Each color channel of the image is represented by a $N \times N$ matrix. Elements of the matrix represent average values of color assigned to pixels/cells.

Apply WENO **interpolation** (step 1...4) in the **x direction** to **subdivide** each cell/pixel. Result is a reconstructed $N \times n\,N$ matrix. Repeat the process in the **y direction** to get the final $n\,N \times n\,N$ matrix.

Reconstruction is applied separately on each color channel of the image.

References

[1] B. Crnković, N. Črnjarić-Žic. Binary weighted essentially non-oscillatory (BWENO) approximation. Journal of Computational and Applied mathematics, 236 (2012)

[2] C. Santaera, R. M. Pidatella, F. Stanco, ENO/WENO Interpolation Methods for the ZOOMING of Digital Images Communications to SIMAI Congress, 1 (2006)