

# EXPONENTIAL DICHOTOMIES WITH RESPECT TO A SEQUENCE OF NORMS AND ADMISSIBILITY

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ABSTRACT. For a nonautonomous dynamics defined by a sequence of linear operators, we introduce the notion of an exponential dichotomy with respect to a sequence of norms and we characterize it completely in terms of the admissibility in  $l^p$  spaces, both for the space of perturbations and the space of solutions. This allows unifying the notions of uniform and nonuniform exponential behavior. Moreover, we consider the general case of a noninvertible dynamics. As a nontrivial application we show that the conditional stability of a nonuniform exponential dichotomy persists under sufficiently small linear perturbations.

## 1. INTRODUCTION

The objective of our paper is to characterize completely a generalization of the notion of an exponential dichotomy in terms of the admissibility in  $l^p$  spaces, both for the space of perturbations and the space of solutions. More precisely, we consider the notion of an exponential dichotomy with respect to a sequence of norms, for a general nonautonomous dynamics defined by a sequence of linear operators. While a uniform exponential behavior corresponds to consider a constant sequence of norms, a nonuniform exponential behavior can be defined in terms of a sequence of Lyapunov norms. Thus, our work unifies the two notions as well as allows considering arbitrary sequences of equivalent norms possibly with unbounded constants. Our approach consists in characterizing the notion of an exponential dichotomy with respect to a sequence of norms in terms of the invertibility of certain operators. This is partly inspired in a related approach of Henry in [12].

The study of admissibility goes back to pioneering work of Perron in [21] and referred originally to the existence of bounded solutions of the equation

$$x' = A(t)x + f(t)$$

in  $\mathbb{R}^n$  for any bounded continuous function  $f: \mathbb{R}_0^+ \rightarrow \mathbb{R}^n$ . This property can be used to deduce the stability or the conditional stability under sufficiently small perturbations of a linear equation.

A relatively simple modification of Perron's work for continuous time yields the following result for discrete time.

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**Theorem 1.** *Let  $(A_m)_{m \in \mathbb{N}}$  be a sequence of  $n \times n$  matrices. If for each bounded sequence  $(f_m)_{m \in \mathbb{N}} \subset \mathbb{R}^n$  there exists  $x_0 \in \mathbb{R}^n$  such that the sequence*

$$x_m = A_{m-1}x_{m-1} + f_m, \quad m \in \mathbb{N} \quad (1)$$

*is bounded, then any bounded sequence  $(A_m \cdots A_1 x)_{m \in \mathbb{N}}$  tends to zero when  $m \rightarrow \infty$ .*

The assumption in Theorem 1 is called the admissibility of the pair of spaces in which we take the perturbation  $(f_m)_{m \in \mathbb{N}}$  and look for the solution  $(x_m)_{m \in \mathbb{N}}$  of equation (1). Up to the fact that Perron only considered continuous time, Theorem 1 is perhaps the first result discussing the relation between admissibility and stability. There is an extensive related literature. For some of the most relevant early contributions in the area we refer to the books by Massera and Schäffer [17] (culminating the development initiated with their paper [16]) and by Dalec'kiĭ and Kreĭn [11]. We also refer to [15] for some early results in infinite-dimensional spaces. For a detailed list of references, we refer to [7] and for more recent work to Huy [13]. We mention in particular the papers [18, 24, 25] as an illustration of various approaches. In [24] the authors consider the problem of the admissibility in  $l^p$  spaces for *uniform* exponential dichotomies assuming in addition that the evolution operator has bounded growth. We emphasize that the bounded growth property fails for any nonuniform exponential dichotomy that is not uniform (as it also fails for many uniform exponential dichotomies). In addition, the admissibility of certain pairs of spaces is related to the invertibility or the Fredholm properties of certain operators (see in particular [5, 6, 14, 20, 26] and the books [7, 10, 11, 12, 17]).

As an application of our results, we give another proof of the robustness of nonuniform exponential dichotomies. For uniform exponential dichotomies, the problem was discussed by Massera and Schäffer [16] (see also [17]), Coppel [9] and in the case of Banach spaces by Dalec'kiĭ and Kreĭn [11], with different approaches and successive generalizations. For more recent works we refer to [8, 19, 22, 23] and the references therein. We note that all these works consider only the case of uniform exponential dichotomies. We refer to [3, 4] for the general case of nonuniform exponential behavior.

A principal motivation to consider the notion of a nonuniform exponential dichotomy is that from the point of view of ergodic theory *almost all* linear variational equations have a nonuniform exponential behavior. More precisely, consider a flow  $(\phi_t)_{t \in \mathbb{R}}$  defined by an autonomous equation  $x' = f(x)$  in  $\mathbb{R}^n$  preserving a finite measure  $\mu$ . This means that  $\mu(\phi_t A) = \mu(A)$  for any measurable set  $A \subset \mathbb{R}^n$  and any  $t \in \mathbb{R}$ . Then the trajectory of  $\mu$ -almost every point  $x$  with nonzero Lyapunov exponents gives rise to a linear variational equation

$$v' = A_x(t)v, \quad \text{with} \quad A_x(t) = d_{\phi_t x} f,$$

admitting a nonuniform exponential dichotomy. We refer to [2, 4] for details.

We refer the reader to [1] for related results in the case of continuous time.

2. CHARACTERIZATION OF DICHOTOMIES VIA  $l^\infty$ -ADMISSIBILITY

In this section we give a complete characterization of the notion of an exponential dichotomy with respect to a sequence of norms in terms of the admissibility in the space  $l^\infty$ .

We first introduce the notion of an exponential dichotomy with respect to a sequence of norms. Let  $X = (X, \|\cdot\|)$  be a Banach space and let  $B(X)$  be the space of all bounded linear operators on  $X$ . Given a sequence  $(A_m)_{m \in \mathbb{Z}}$  in  $B(X)$ , we define

$$\mathcal{A}(n, m) = \begin{cases} A_{n-1} \cdots A_m & \text{if } n > m, \\ \text{Id} & \text{if } n = m. \end{cases}$$

We say that  $(A_m)_{m \in \mathbb{Z}}$  admits an *exponential dichotomy* with respect to a sequence of norms  $\|\cdot\|_m$  if:

1. there exist projections  $P_m: X \rightarrow X$  for each  $m \in \mathbb{Z}$  satisfying

$$A_m P_m = P_{m+1} A_m \quad \text{for } m \in \mathbb{Z} \quad (2)$$

such that each map  $A_m|_{\ker P_m}: \ker P_m \rightarrow \ker P_{m+1}$  is invertible;

2. there exist constants  $\lambda, D > 0$  such that for each  $x \in X$  and  $n, m \in \mathbb{Z}$  we have

$$\|\mathcal{A}(n, m) P_m x\|_n \leq D e^{-\lambda(n-m)} \|x\|_m \quad \text{for } n \geq m \quad (3)$$

and

$$\|\mathcal{A}(n, m) Q_m x\|_n \leq D e^{-\lambda(m-n)} \|x\|_m \quad \text{for } n \leq m, \quad (4)$$

where  $Q_m = \text{Id} - P_m$  and where

$$\mathcal{A}(n, m) = (\mathcal{A}(m, n)|_{\ker P_n})^{-1} : \ker P_m \rightarrow \ker P_n$$

for  $n < m$ .

We always assume in the paper that each norm  $\|\cdot\|_m$  is equivalent to the original norm  $\|\cdot\|$ .

Let  $Y$  be the set of all sequences  $\mathbf{x} = (x_m)_{m \in \mathbb{Z}}$ ,  $x_m \in X$  such that

$$\|\mathbf{x}\|_\infty := \sup_{m \in \mathbb{Z}} \|x_m\|_m < +\infty.$$

It is easy to verify that  $Y = (Y, \|\cdot\|_\infty)$  is a Banach space.

Our first result shows that the existence of an exponential dichotomy with respect to a sequence of norms implies  $l^\infty$ -admissibility.

**Theorem 2.** *If the sequence  $(A_m)_{m \in \mathbb{Z}}$  admits an exponential dichotomy with respect to the sequence of norms  $\|\cdot\|_m$ , then for each  $\mathbf{y} \in Y$  there exists a unique  $\mathbf{x} \in Y$  such that*

$$x_n - A_{n-1} x_{n-1} = y_n \quad (5)$$

for  $n \in \mathbb{Z}$ .

*Proof.* Take a sequence  $\mathbf{y} = (y_n)_{n \in \mathbb{Z}} \in Y$ . For each  $n \in \mathbb{Z}$ , let

$$x_n^1 = \sum_{m \geq 0} \mathcal{A}(n, n-m) P_{n-m} y_{n-m} \quad (6)$$

and

$$x_n^2 = - \sum_{m \geq 1} \mathcal{A}(n, n+m) Q_{n+m} y_{n+m}. \quad (7)$$

We have

$$\|x_n^1\|_n \leq \sum_{m \geq 0} D e^{-\lambda m} \|y_{n-m}\|_{n-m} \leq \frac{D}{1 - e^{-\lambda}} \|\mathbf{y}\|_\infty$$

and similarly,

$$\|x_n^2\|_n \leq \frac{D e^{-\lambda}}{1 - e^{-\lambda}} \|\mathbf{y}\|_\infty.$$

For each  $n \in \mathbb{Z}$ , we define  $x_n = x_n^1 + x_n^2$  and let  $\mathbf{x} = (x_n)_{n \in \mathbb{Z}}$ . It follows from the inequalities that  $\mathbf{x} \in Y$ . Furthermore, it is easy to verify that (5) holds for  $n \in \mathbb{Z}$ .

Now we establish the uniqueness of  $\mathbf{x}$ . Since the map  $\mathbf{x} \mapsto \mathbf{y}$  defined by (5) is linear, it is sufficient to show that if  $x_n = A_{n-1} x_{n-1}$  for  $n \in \mathbb{Z}$ , with  $\mathbf{x} = (x_m)_{m \in \mathbb{Z}} \in Y$ , then  $x_n = 0$  for  $n \in \mathbb{Z}$ . Let

$$x_n^s = P_n x_n \quad \text{and} \quad x_n^u = Q_n x_n.$$

Then  $x_n = x_n^s + x_n^u$  and it follows from (2) that

$$x_n^s = A_{n-1} x_{n-1}^s \quad \text{and} \quad x_n^u = A_{n-1} x_{n-1}^u$$

for  $n \in \mathbb{Z}$ . Since  $x_k^s = \mathcal{A}(k, k-m) x_{k-m}^s$  for  $m \geq 0$ , we have

$$\begin{aligned} \|x_k^s\|_k &= \|\mathcal{A}(k, k-m) x_{k-m}^s\|_k \\ &= \|\mathcal{A}(k, k-m) Q_{k-m} x_{k-m}\|_k \\ &\leq D e^{-\lambda m} \|x_{k-m}\|_{k-m} \\ &\leq D e^{-\lambda m} \|\mathbf{x}\|_\infty. \end{aligned} \quad (8)$$

Letting  $m \rightarrow +\infty$  yields that  $x_k^s = 0$  for  $k \in \mathbb{Z}$ . Similarly, since  $x_k^u = \mathcal{A}(k, k+m) x_{k+m}^u$  for  $m \geq 0$ , we have

$$\begin{aligned} \|x_k^u\|_k &= \|\mathcal{A}(k, k+m) x_{k+m}^u\|_k \\ &= \|\mathcal{A}(k, k+m) P_{k+m} x_{k+m}\|_k \\ &\leq D e^{-\lambda m} \|x_{k+m}\|_{k+m} \\ &\leq D e^{-\lambda m} \|\mathbf{x}\|_\infty. \end{aligned} \quad (9)$$

This implies  $x_k^u = 0$  for  $k \in \mathbb{Z}$  and hence  $x_m = 0$  for  $m \in \mathbb{Z}$ .  $\square$

Now we establish the converse of Theorem 2. We note that the proof is much more technical in comparison to that of Theorem 2.

**Theorem 3.** *Assume that for each  $\mathbf{y} \in Y$  there exists a unique  $\mathbf{x} \in Y$  such that (5) holds for  $n \in \mathbb{Z}$ . Then the sequence  $(A_m)_{m \in \mathbb{Z}}$  admits an exponential dichotomy with respect to the sequence of norms  $\|\cdot\|_m$ .*

*Proof.* We divide the proof into steps.

*Step 1. Construction of auxiliary operators.* Let  $T: \mathcal{D}(T) \subset Y \rightarrow Y$  be the linear operator defined by

$$(T\mathbf{x})_n = x_n - A_{n-1}x_{n-1} \quad (10)$$

in the domain  $\mathcal{D}(T)$  formed by all  $\mathbf{x} \in Y$  such that  $T\mathbf{x} \in Y$ .

**Lemma 1.** *The operator  $T: \mathcal{D}(T) \subset Y \rightarrow Y$  is closed.*

*Proof of the lemma.* Assume that  $(\mathbf{x}^k)_{k \in \mathbb{N}}$  is a sequence in  $\mathcal{D}(T)$  converging to  $\mathbf{x} \in Y$  and that  $T\mathbf{x}^k$  converges to  $\mathbf{y} \in Y$ . Then

$$x_n - A_{n-1}x_{n-1} = \lim_{k \rightarrow +\infty} (x_n^k - A_{n-1}x_{n-1}^k) = \lim_{k \rightarrow +\infty} (T\mathbf{x}^k)_n = y_n$$

for  $n \in \mathbb{Z}$ . We conclude that  $\mathbf{x} \in \mathcal{D}(T)$  and that  $T\mathbf{x} = \mathbf{y}$ . Therefore, the operator  $T$  is closed.  $\square$

It follows from the assumptions in the theorem that  $T$  is invertible. We claim that  $G = T^{-1}: Y \rightarrow Y$  has a closed graph in  $Y \times Y$ . Indeed, let  $(\mathbf{y}^k)_{k \in \mathbb{N}}$  be a sequence in  $Y$  converging to  $\mathbf{y} \in Y$  and assume that  $\mathbf{x}^k = G\mathbf{y}^k$  converges to  $\mathbf{x} \in Y$ . Then the sequence  $(\mathbf{x}^k, T\mathbf{x}^k) = (\mathbf{x}^k, \mathbf{y}^k)$ , which is in the graph of  $T$ , converges to  $(\mathbf{x}, \mathbf{y})$ . By Lemma 1, the operator  $T$  is closed and hence  $\mathbf{x} \in \mathcal{D}(T)$  and  $\mathbf{y} = T\mathbf{x}$ . Therefore,

$$(\mathbf{y}^k, G\mathbf{y}^k) \rightarrow (\mathbf{y}, \mathbf{x}) = (\mathbf{y}, G\mathbf{y})$$

when  $k \rightarrow \infty$ , which shows that  $G$  is closed. It follows from the closed graph theorem that  $G$  is bounded.

For each  $n, k \in \mathbb{Z}$  we define a linear operator  $G_{n,k}: X \rightarrow X$  as follows. Take  $y \in X$  and define  $\mathbf{y} = (y_m)_{m \in \mathbb{Z}} \in Y$  by  $y_k = y$  and  $y_m = 0$  for  $m \neq k$ . By hypothesis, there exists a unique  $\mathbf{x} \in Y$  such that  $T\mathbf{x} = \mathbf{y}$  and we set  $G_{n,k}y = x_n$ . We note that

$$\begin{aligned} \|G_{n,k}y\|_n &= \|x_n\|_n \leq \|\mathbf{x}\|_\infty = \|G\mathbf{y}\|_\infty \\ &\leq \|G\| \cdot \|\mathbf{y}\|_\infty = \|G\| \cdot \|y\|_k \end{aligned}$$

for  $y \in X$  and that

$$G_{n,k} - A_{n-1}G_{n-1,k} = \begin{cases} \text{Id} & \text{if } n = k, \\ 0 & \text{if } n \neq k \end{cases} \quad (11)$$

for  $n, k \in \mathbb{Z}$ . Indeed,

$$G_{n,k}y - A_{n-1}G_{n-1,k}y = x_n - A_{n-1}x_{n-1} = y_n = \begin{cases} y & \text{if } n = k, \\ 0 & \text{if } n \neq k. \end{cases}$$

We also establish a formula for the operator  $G$ .

**Lemma 2.** *We have*

$$(G\mathbf{y})_n = \sum_{k=-\infty}^{+\infty} G_{n,k}y_k, \quad n \in \mathbb{Z} \quad (12)$$

for all  $\mathbf{y} \in Y$  such that  $y_k = 0$  for any sufficiently large  $|k|$ .

*Proof of the lemma.* Take  $\mathbf{y} \in Y$  such that  $y_k = 0$  for  $|k| > l$ , for some  $l \in \mathbb{N}$ . For every  $k \in \mathbb{Z}$  with  $|k| \leq l$  we define  $\mathbf{y}^k = (y_m^k)_{m \in \mathbb{Z}} \in Y$  by

$$y_m^k = \begin{cases} y_k & \text{if } m = k, \\ 0 & \text{if } m \neq k. \end{cases}$$

Clearly,  $\mathbf{y} = \sum_{k=-l}^l \mathbf{y}^k$ . For each  $k \in \mathbb{Z}$  with  $|k| \leq l$  take  $\mathbf{x}^k \in Y$  such that  $T\mathbf{x}^k = \mathbf{y}^k$  and let  $\mathbf{x} = \sum_{k=-l}^l \mathbf{x}^k$ . Then  $\mathbf{x} \in Y$  and  $T\mathbf{x} = \mathbf{y}$ . Hence,

$$(G\mathbf{y})_n = x_n = \sum_{k=-l}^l x_n^k = \sum_{k=-l}^l G_{n,k} y_k = \sum_{k=-\infty}^{+\infty} G_{n,k} y_k$$

for  $n \in \mathbb{Z}$ .  $\square$

*Step 2. Construction of projections and their invariance.* Consider the maps  $P_m = G_{m,m}$ . Using (11) it follows by induction that

$$G_{n,m} = \mathcal{A}(n,m)P_m \quad \text{for } n \geq m \quad (13)$$

and

$$\mathcal{A}(m,n)G_{n,m} = -Q_m \quad \text{for } n < m, \quad (14)$$

where  $Q_m = \text{Id} - P_m$ .

**Lemma 3.** *If the sequence  $x_{n+1} = A_n x_n$ ,  $n \geq m$  is bounded, that is,  $\sup_{n \geq m} \|x_n\|_n < +\infty$ , then  $Q_m x_m = 0$ .*

*Proof of the lemma.* Let  $x_n = 0$  for  $n < m$ . Then  $\mathbf{x} = (x_n)_{n \in \mathbb{Z}} \in Y$  and

$$x_n - A_{n-1} x_{n-1} = \begin{cases} x_m & \text{if } n = m, \\ 0 & \text{if } n \neq m \end{cases}$$

for  $n \in \mathbb{Z}$ . Therefore,  $x_n = G_{n,m} x_m$  for  $n \in \mathbb{Z}$  and in particular

$$x_m = G_{m,m} x_m = P_m x_m.$$

Hence  $Q_m x_m = 0$ .  $\square$

Now we use Lemma 3 to show that each map  $P_m$  is a projection. Take  $x \in X$  and let  $x_n = G_{n,m} x$ . Then  $x_{n+1} = A_n x_n$  for  $n \geq m$  and the sequence  $(x_n)_{n \in \mathbb{Z}}$  is bounded. It follows from Lemma 3 that  $Q_m x_m = 0$ . Therefore,

$$0 = Q_m G_{m,m} x = (\text{Id} - P_m) P_m x.$$

Hence,  $P_m x = P_m^2 x$  for  $x \in X$ , which shows that  $P_m$  is a projection. We also note that if  $Q_m x = 0$ , then  $x_m = x$ . Since  $P_{m+1} x_{m+1} = x_{m+1}$ , we have

$$Q_{m+1} A_m x = A_m Q_m x \quad \text{when } Q_m x = 0. \quad (15)$$

**Lemma 4.** *If  $(x_n)_{n \leq m}$  is a bounded sequence such that  $x_{n+1} = A_n x_n$  for  $n < m$ , then  $A_m x_m \in \ker P_{m+1}$ .*

*Proof of the lemma.* Let  $x_n = 0$  for  $n > m$ . Then  $\mathbf{x} = (x_n)_{n \in \mathbb{Z}} \in Y$  and  $x_n = -G_{n,m+1} A_m x_m$  for  $n \in \mathbb{Z}$ . In particular,

$$0 = x_{m+1} = -P_{m+1} A_m x_m$$

and the lemma follows.  $\square$

Take  $x \in \ker P_m$  and let  $y_n = G_{n,m}x$ . Then the sequence  $(y_n)_{n \in \mathbb{Z}}$  is bounded and  $y_{n+1} = A_n y_n$  for  $n < m - 1$ . It follows from Lemma 4 that  $A_{m-2}y_{m-2} = y_{m-1} \in \ker P_{m-1}$ . Since  $y_m = P_m x = 0$  we have

$$0 = G_{m,m}x = (A_{m-1}G_{m-1,m} + \text{Id})x.$$

This implies that

$$-x = A_{m-1}G_{m-1,m}x = A_{m-1}y_{m-1} \in A_{m-1} \ker P_{m-1}.$$

Hence,  $\ker P_m \subset A_{m-1} \ker P_{m-1}$ . Now let  $\bar{y}_n = y_n$  for  $n < m$  and  $\bar{y}_n = A_{n-1}\bar{y}_{n-1}$  for  $n \geq m$ . Then

$$\bar{y}_m = A_{m-1}y_{m-1} = -x$$

and the sequence  $(\bar{y}_n)_{n \leq m}$  is bounded. Again, it follows from Lemma 4 that  $A_m x \in \ker P_{m+1}$ . Hence,  $A_m \ker P_m \subset \ker P_{m+1}$ . We conclude that

$$A_m \ker P_m = \ker P_{m+1} \quad \text{for } m \in \mathbb{Z}.$$

Finally, we show that the map

$$A_m | \ker P_m : \ker P_m \rightarrow \ker P_{m+1} \tag{16}$$

is an isomorphism. It is sufficient to prove that it is injective. Assume that  $A_m x = 0$  for  $x \in \ker P_m$ . Then  $\bar{y}_n = 0$  for  $n > m$ . It follows from Lemma 3 that  $Q_m \bar{y}_m = 0$  and thus  $x = Q_m x = 0$ .

In particular, if  $x \in \ker P_m = \text{Im } Q_m$ , then  $A_m x \in \ker P_{m+1} = \text{Im } Q_{m+1}$  and thus  $Q_{m+1} A_m x = A_m Q_m x$ . Since the equality also holds when  $Q_m x = 0$  (see (15)), we conclude that

$$Q_{m+1} A_m = A_m Q_m \quad \text{for } m \in \mathbb{Z}.$$

*Step 3. Norm bounds.* In order to show that the sequence  $(A_m)_{m \in \mathbb{Z}}$  admits an exponential dichotomy, it remains to prove that (3) and (4) hold for some constants  $D, \lambda > 0$ . Take  $x \in X$  and  $n \geq m$ . We assume that  $\mathcal{A}(n, m)P_m x \neq 0$ . Then

$$\phi_k^{-1} := \|\mathcal{A}(k, m)P_m x\|_k > 0 \quad \text{for } m \leq k \leq n.$$

We have

$$\sum_{k=m}^n \mathcal{A}(n, k)P_k \mathcal{A}(k, m)P_m x \phi_k = \mathcal{A}(n, m)P_m x \sum_{k=m}^n \phi_k,$$

which can be rewritten in the form

$$\sum_{k=m}^n G_{n,k} \mathcal{A}(k, m)P_m x \phi_k = \mathcal{A}(n, m)P_m x \sum_{k=m}^n \phi_k.$$

Now let

$$y_k = \phi_k \mathcal{A}(k, m)P_m x \quad \text{for } m \leq k \leq n$$

and  $y_k = 0$  otherwise. Then  $\mathbf{y} = (y_k)_{k \in \mathbb{Z}} \in Y$  and  $\|\mathbf{y}\|_\infty = 1$ . It follows from Lemma 2 that

$$(G\mathbf{y})_n = \sum_{k=m}^n G_{n,k} \mathcal{A}(k, m)P_m x \phi_k$$

and hence,

$$\begin{aligned}\phi_n^{-1} \sum_{k=m}^n \phi_k &= \left\| \mathcal{A}(n, m) P_m x \sum_{k=m}^n \phi_k \right\|_n \\ &= \|(G\mathbf{y})_n\|_n \leq \|G\mathbf{y}\|_\infty \leq \|G\|.\end{aligned}$$

In particular,

$$\|G\| \geq \phi_n^{-1} \sum_{k=m}^n \phi_k \geq \phi_n^{-1} \phi_n = 1$$

and in fact we conclude that  $\|G\| > 1$  taking  $m < n$ . From now we assume that  $m < n$ . Let  $\psi_n = \sum_{k=m}^n \phi_k$ . Then  $\psi_{n-1} \leq (1 - \|G\|^{-1})\psi_n$  and

$$\phi_n \geq \|G\|^{-1}(1 - \|G\|^{-1})^{m-n} \phi_m.$$

Therefore,

$$\begin{aligned}\|\mathcal{A}(n, m) P_m x\|_n &\leq \|G\|(1 - \|G\|^{-1})^{n-m} \|P_m x\|_m \\ &= \|G\|(1 - \|G\|^{-1})^{n-m} \|G_{m,m} x\|_m \\ &\leq \|G\|^2(1 - \|G\|^{-1})^{n-m} \|x\|_m,\end{aligned}\tag{17}$$

provided that  $\mathcal{A}(n, m) P_m x \neq 0$  and  $m < n$ . Moreover, the inequality holds trivially when  $\mathcal{A}(n, m) P_m x = 0$  or  $m = n$ .

Now take  $x \in X$  and  $n < m$  such that  $\mathcal{A}(n, m) Q_m x \neq 0$ . Then,

$$\rho_k^{-1} := \|\mathcal{A}(k, m) Q_m x\|_k > 0 \quad \text{for } n < k \leq m.$$

We have

$$\sum_{k=n+1}^m \mathcal{A}(n, k) Q_k \mathcal{A}(k, m) Q_m x \rho_k = \mathcal{A}(n, m) Q_m x \sum_{k=n+1}^m \rho_k$$

which can be rewritten in the form

$$- \sum_{k=n+1}^m G_{n,k} \mathcal{A}(k, m) Q_m x \rho_k = \mathcal{A}(n, m) Q_m x \sum_{k=n+1}^m \rho_k.$$

Proceeding in a similar manner to that for the sequence  $\phi_m$  we obtain

$$\rho_n^{-1} \sum_{k=n+1}^m \rho_k \leq \|G\|.$$

Let  $\sigma_n = \sum_{k=n+1}^m \rho_k$ . Then  $(1 + \|G\|^{-1})\sigma_n \leq \sigma_{n-1}$  and

$$\|G\|^{-1}(1 + \|G\|^{-1})^{m-n-1} \rho_m \leq \rho_n.$$

Therefore, by (17), we obtain

$$\begin{aligned}\|\mathcal{A}(n, m) Q_m x\|_n &\leq \|G\|(1 + \|G\|^{-1})^{n-m+1} \|Q_m x\|_m \\ &\leq \|G\|(1 + \|G\|^{-1})^{n-m+1} (1 + \|G\|) \|x\|_m \\ &\leq (1 + \|G\|)^2 (\|G\|/1 + \|G\|)^{m-n} \|x\|_m.\end{aligned}$$

This completes the proof of the theorem.  $\square$



3. CHARACTERIZATION OF DICHOTOMIES VIA  $l^p$ -ADMISSIBILITY

In this section we obtain corresponding results to those in Section 2 for  $l^p$  spaces obtained from a sequence of norms, for  $p < +\infty$ . Namely, we give a complete characterization of the notion of an exponential dichotomy with respect to a sequence of norms in terms of the admissibility in the space  $l^p$ .

Take  $1 \leq p < +\infty$ . Let  $Y$  be the set of all sequences  $\mathbf{x} = (x_m)_{m \in \mathbb{Z}}$ ,  $x_m \in X$  such that

$$\|\mathbf{x}\|_p := \left( \sum_{m=-\infty}^{\infty} \|x_m\|_m^p \right)^{1/p} < +\infty.$$

It is easy to verify that  $Y = (Y, \|\cdot\|_p)$  is a Banach space. We recall that if  $\mathbf{a} = (a_n)_{n \in \mathbb{Z}}$  and  $\mathbf{b} = (b_n)_{n \in \mathbb{Z}}$  are two sequences of real numbers, then their convolution is the sequence  $\mathbf{a} \star \mathbf{b}$  defined by

$$(\mathbf{a} \star \mathbf{b})_n = \sum_{m=-\infty}^{\infty} a_m b_{n-m}, \quad n \in \mathbb{Z}$$

and it satisfies Young's inequality: if  $\mathbf{a} \in l^p$ ,  $\mathbf{b} \in l^q$  and

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

with  $1 \leq p, q, r \leq +\infty$ , then  $\mathbf{a} \star \mathbf{b} \in l^r$  and

$$\|\mathbf{a} \star \mathbf{b}\|_r \leq \|\mathbf{a}\|_p \cdot \|\mathbf{b}\|_q. \quad (18)$$

The following is a version of Theorem 2 for  $l^p$  spaces with  $p < +\infty$ .

**Theorem 4.** *If the sequence  $(A_m)_{m \in \mathbb{Z}}$  admits an exponential dichotomy with respect to the sequence of norms  $\|\cdot\|_m$ , then for every  $\mathbf{y} \in Y$  there exists a unique  $\mathbf{x} \in Y$  such that (5) holds for  $n \in \mathbb{Z}$ .*

*Proof.* Take a sequence  $\mathbf{y} = (y_n)_{n \in \mathbb{Z}} \in Y$ . For each  $n \in \mathbb{Z}$ , let  $x_n^1$  and  $x_n^2$  be as in (6) and (7).

**Lemma 5.** *The sequences  $\mathbf{x}^1 = (x_n^1)_{n \in \mathbb{Z}}$  and  $\mathbf{x}^2 = (x_n^2)_{n \in \mathbb{Z}}$  belong to  $Y$ .*

*Proof of the lemma.* We have

$$\|x_n^1\|_n \leq \sum_{m \geq 0} D e^{-\lambda m} \|y_{n-m}\|_{n-m}$$

for  $n \in \mathbb{Z}$ . Let

$$a_n = \begin{cases} e^{-\lambda n} & \text{if } n \geq 0, \\ 0 & \text{if } n < 0 \end{cases} \quad \text{and} \quad b_n = \|y_n\|_n$$

for  $n \in \mathbb{Z}$ . Clearly,  $\mathbf{a} = (a_n)_{n \in \mathbb{Z}} \in l^\infty$  and  $\mathbf{b} = (b_n)_{n \in \mathbb{Z}} \in l^p$ . It follows from (18) that  $\mathbf{a} \star \mathbf{b} \in l^p$  and hence  $(\|x_n^1\|_n)_{n \in \mathbb{Z}} \in l^p$ . Therefore,  $\mathbf{x}^1 \in Y$ . One can show in a similar manner that  $\mathbf{x}^2 \in Y$ .  $\square$

It follows from Lemma 5 that  $\mathbf{x} = \mathbf{x}^1 + \mathbf{x}^2 \in Y$ . The remainder of the proof is identical to that of Theorem 2, simply replacing  $\|\mathbf{x}\|_\infty$  by  $\|\mathbf{x}\|_p$  in (8) and (9).  $\square$

Now we establish the converse of Theorem 4. We emphasize that a large part of the argument substantially differs from that in the proof of Theorem 3, reason for which we separated the two results.

**Theorem 5.** *Assume that for each  $\mathbf{y} \in Y$  there exists a unique  $\mathbf{x} \in Y$  such that (5) holds for  $n \in \mathbb{Z}$ . Then the sequence  $(A_m)_{m \in \mathbb{Z}}$  admits an exponential dichotomy with respect to the sequence of norms  $\|\cdot\|_m$ .*

*Proof.* The first part of the proof has many similarities to Steps 1 and 2 in the proof of Theorem 3 and so we only sketch the argument.

Let  $T$  be the closed linear operator defined by (10) in the domain  $\mathcal{D}(T)$  formed by all  $\mathbf{x} \in Y$  such that  $T\mathbf{x} \in Y$ . As in the proof of Theorem 3, we also consider its inverse  $G: Y \rightarrow Y$ , which is bounded, and the operators  $G_{n,k}: X \rightarrow X$  for  $n, k \in \mathbb{Z}$ . We note that

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \|G_{n,k}y\|_n^p &= \sum_{n=-\infty}^{\infty} \|x_n\|_n^p = \|\mathbf{x}\|_p^p = \|G\mathbf{y}\|_p^p \\ &\leq \|G\|^p \cdot \|\mathbf{y}\|_p^p = \|G\|^p \cdot \|y\|_k^p \end{aligned}$$

for  $y \in X$ . Moreover, the identities in (11) and (12) hold in the present context and their proofs are identical.

Now we consider the maps  $P_m = G_{m,m}$ . Repeating the proof of Lemma 3 yields the following result.

**Lemma 6.** *If the sequence defined by  $x_{n+1} = A_n x_n$ , for  $n \geq m$ , satisfies  $\sum_{n=m}^{\infty} \|x_n\|_n^p < +\infty$ , then  $Q_m x_m = 0$ .*

Now we show that each map  $P_m$  is a projection. Take  $x \in X$  and let  $x_n = G_{n,m}x$ . Then  $x_{n+1} = A_n x_n$  for  $n \geq m$  and

$$\sum_{n=m}^{\infty} \|x_n\|_n^p \leq \sum_{n=-\infty}^{\infty} \|x_n\|_n^p < +\infty.$$

It follows from Lemma 6 that  $Q_m x_m = 0$ . This implies that

$$0 = Q_m G_{m,m}x = (\text{Id} - P_m)P_m x.$$

Hence,  $P_m x = P_m^2 x$  for  $x \in X$ , which shows that  $P_m$  is a projection. We also note that if  $Q_m x = 0$ , then  $x_m = x$ . Since  $P_{m+1}x_{m+1} = x_{m+1}$  we have

$$Q_{m+1}A_m x = A_m Q_m x \quad \text{when} \quad Q_m x = 0. \quad (19)$$

Similarly, repeating the proof of Lemma 4 yields the following result.

**Lemma 7.** *If  $(x_n)_{n \leq m}$  is a sequence such that  $\sum_{n=-\infty}^m \|x_n\|_n^p < +\infty$  and  $x_{n+1} = A_n x_n$  for  $n < m$ , then  $A_m x_m \in \ker P_{m+1}$ .*

Take  $x \in \ker P_m$  and let  $y_n = G_{n,m}x$ . Then  $\sum_{n=-\infty}^{\infty} \|y_n\|_n^p < +\infty$  and  $y_{n+1} = A_n y_n$  for  $n < m-1$ . It follows from Lemma 7 that  $A_{m-2}y_{m-2} = y_{m-1} \in \ker P_{m-1}$ . Since  $y_m = P_m x = 0$  we have

$$0 = G_{m,m}x = (A_{m-1}G_{m-1,m} + \text{Id})x.$$

This implies that

$$-x = A_{m-1}G_{m-1,m}x = A_{m-1}y_{m-1} \in A_{m-1} \ker P_{m-1}.$$

Hence,  $\ker P_m \subset A_{m-1} \ker P_{m-1}$ . Now let  $\bar{y}_n = y_n$  for  $n < m$  and  $\bar{y}_n = A_{n-1} \bar{y}_{n-1}$  for  $n \geq m$ . Then

$$\bar{y}_m = A_{m-1} y_{m-1} = -x$$

and  $\sum_{n=-\infty}^m \|\bar{y}_n\|_n^p < +\infty$ . It follows from Lemma 7 that  $A_m x \in \ker P_{m+1}$ . Hence,  $A_m \ker P_m \subset \ker P_{m+1}$ . We conclude that

$$A_m \ker P_m = \ker P_{m+1} \quad \text{for } m \in \mathbb{Z}.$$

Together with (19) this implies that

$$P_{m+1} A_m = A_m P_m \quad \text{for } m \in \mathbb{Z}.$$

One can now proceed as in the proof of Theorem 3 to show that the map  $A_m|_{\ker P_m}$  in (16) is an isomorphism, with the role of Lemma 3 now played by Lemma 6.

It remains to show that (3) and (4) hold for some constants  $D, \lambda > 0$ . This is the part of the proof that substantially differs from that of Theorem 3 (see Step 3).

We start with an estimate along vectors in  $\text{Im } P_l$ .

**Lemma 8.** *There exist constants  $D, \lambda > 0$  such that*

$$\|\mathcal{A}(k, l) P_l x\|_k \leq D e^{-\lambda(k-l)} \|x\|_l \quad (20)$$

for  $x \in X$  and  $k \geq l$ .

*Proof of the lemma.* Take  $n, m \in \mathbb{Z}$  with  $n \geq m$  and  $x \in \text{Im } P_m$ . We define

$$x_k = \begin{cases} \mathcal{A}(k, m)x & \text{if } k \geq m, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad y_k = \begin{cases} x & \text{if } k = m, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly,  $\mathbf{y} = (y_k)_{k \in \mathbb{Z}} \in Y$ . Moreover, by (13), we have

$$x_k = \mathcal{A}(k, m)x = \mathcal{A}(k, m)P_m x = G_{k,m}x$$

for  $k \geq m$ . It follows from the definition of  $G_{k,m}$  that  $(G_{k,m}x)_{k \in \mathbb{Z}}$  belongs to  $Y$  and so does  $\mathbf{x} = (x_k)_{k \in \mathbb{Z}}$ . Moreover,  $T\mathbf{x} = \mathbf{y}$ . Hence, one can write

$$\begin{aligned} \|G\| \cdot \|x\|_m &= \|G\| \cdot \|\mathbf{y}\|_p \geq \|\mathbf{x}\|_p \\ &\geq \|x_n\|_n = \|\mathcal{A}(n, m)x\|_n. \end{aligned} \quad (21)$$

Now take  $n \in \mathbb{N}$  such that

$$n + 1 \geq (e\|G\|^4)^{p/(p-1)} \quad (22)$$

and  $m \in \mathbb{Z}$ . Let  $\phi_k = \|\mathcal{A}(k, m)P_m x\|_k^{-1}$  for  $m \leq k \leq m + n$ . Moreover, we consider the sequence  $\mathbf{y} = (y_k)_{k \in \mathbb{Z}}$  defined by

$$y_k = \begin{cases} \mathcal{A}(k, m)P_m x \phi_k & \text{if } m \leq k \leq m + n, \\ 0 & \text{otherwise.} \end{cases} \quad (23)$$

By (21) we obtain

$$\begin{aligned}\phi_{n+m}^{-1} \sum_{k=m}^{n+m} \phi_k &= \phi_{n+m}^{-1} \sum_{k=m}^{n+m} \frac{1}{\|\mathcal{A}(k, m)P_m x\|_k} \\ &\geq \phi_{n+m}^{-1} \sum_{k=m}^{n+m} \frac{1}{\|G\| \cdot \|P_m x\|_m} \\ &\geq \frac{\phi_{n+m}^{-1}}{\|G\|^2 \|x\|_m} (n+1).\end{aligned}$$

Hence,

$$\begin{aligned}\|\mathcal{A}(m+2n, m)P_m x\|_{m+2n} &= \|\mathcal{A}(m+2n, n+m)\mathcal{A}(n+m, m)P_m x\|_{m+2n} \\ &\leq \|G\| \cdot \|\mathcal{A}(n+m, m)P_m x\|_{n+m} \\ &= \|G\| \phi_{n+m}^{-1} \\ &\leq \frac{\|G\|^3 \|x\|_m}{n+1} \phi_{n+m}^{-1} \sum_{k=m}^{n+m} \phi_k.\end{aligned}\tag{24}$$

On the other hand,

$$\sum_{k=m}^{n+m} \mathcal{A}(n+m, k)P_k \mathcal{A}(k, m)P_m x \phi_k = \mathcal{A}(n+m, m)P_m x \sum_{k=m}^{n+m} \phi_k,$$

which can be rewritten in the form

$$\sum_{k=m}^{n+m} G_{n+m, k} \mathcal{A}(k, m)P_m x \phi_k = \mathcal{A}(n+m, m)P_m x \sum_{k=m}^{n+m} \phi_k.\tag{25}$$

For the sequence  $\mathbf{y}$  defined by (23), it follows from (12) that

$$(G\mathbf{y})_{n+m} = \sum_{k=m}^{n+m} G_{n+m, k} \mathcal{A}(k, m)P_m x \phi_k.$$

Hence, by (25),

$$\begin{aligned}\phi_{n+m}^{-1} \sum_{k=m}^{n+m} \phi_k &= \|(G\mathbf{y})_{n+m}\|_{n+m} \leq \|G\mathbf{y}\|_p \\ &\leq \|G\| \cdot \|\mathbf{y}\|_p = \|G\| (n+1)^{1/p}\end{aligned}$$

and it follows from (24) that

$$\|\mathcal{A}(m+2n, m)P_m x\|_{m+2n} \leq \|G\|^4 \|x\|_m (n+1)^{1/p-1}.$$

By (22) we obtain

$$\|\mathcal{A}(m+2n, m)P_m x\|_{m+2n} \leq \frac{1}{e} \|x\|_m.\tag{26}$$

Now take  $k \geq l$  and write  $k - l$  in the form  $k - l = t \cdot 2n + r$ , with  $t \in \mathbb{N}$  and  $0 \leq r < 2n$ . It follows from (21) and (26) that

$$\begin{aligned} \|\mathcal{A}(k, l)P_l x\|_k &= \|\mathcal{A}(l + 2nt + r, l)P_l x\|_{l+2nt+r} \\ &\leq \frac{1}{e^t} \|\mathcal{A}(l + r, l)P_l x\|_{l+r} \\ &\leq \frac{\|G\|}{e^t} \|P_l x\|_l \\ &\leq 2\|G\|^2 e^{-(k-l)/(2n)} \|x\|_l. \end{aligned}$$

We conclude that (20) holds taking  $D = 2\|G\|^2$  and  $\lambda = 1/(2n)$ .  $\square$

Now we obtain a corresponding estimate along vectors in  $\text{Im } Q_l$ .

**Lemma 9.** *There exist constants  $D, \lambda > 0$  such that*

$$\|\mathcal{A}(k, l)Q_l x\|_k \leq D e^{-\lambda(l-k)} \|x\|_l \quad (27)$$

for  $x \in X$  and  $k \leq l$ .

*Proof of the lemma.* Take  $n, m \in \mathbb{Z}$  with  $n \leq m$  and  $x \in \text{Im } Q_m$ . We define

$$x_k = \begin{cases} \mathcal{A}(k, m)x & \text{if } k < m, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad y_k = \begin{cases} -x & \text{if } k = m, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly,  $\mathbf{y} = (y_k)_{k \in \mathbb{Z}} \in Y$ . Moreover, by (14), we have

$$x_k = \mathcal{A}(k, m)x = \mathcal{A}(k, m)Q_m x = -G_{k,m}x$$

for  $k < m$ . It follows from the definition of  $G_{k,m}$  that  $(G_{k,m}x)_{k \in \mathbb{Z}}$  belongs to  $Y$  and so does  $\mathbf{x} = (x_k)_{k \in \mathbb{Z}}$ . Moreover,  $T\mathbf{x} = \mathbf{y}$ . Hence,

$$\begin{aligned} \|G\| \cdot \|x\|_m &= \|G\| \cdot \|\mathbf{y}\|_p \geq \|\mathbf{x}\|_p \\ &\geq \|x_n\|_n = \|\mathcal{A}(n, m)x\|_n. \end{aligned} \quad (28)$$

Now take  $n \in \mathbb{N}$  such that

$$n \geq (e\|G\|^4)^{p/(p-1)} \quad (29)$$

and  $m \in \mathbb{Z}$ . Let  $\phi_k = \|\mathcal{A}(k, m)Q_m x\|_k^{-1}$  for  $m - n + 1 \leq k \leq m$ . Moreover, we consider the sequence  $\mathbf{y} = (y_k)_{k \in \mathbb{Z}}$  defined by

$$y_k = \begin{cases} \mathcal{A}(k, m)Q_m x \phi_k & \text{if } m - n + 1 \leq k \leq m, \\ 0 & \text{otherwise.} \end{cases} \quad (30)$$

By (28) we obtain

$$\begin{aligned} \phi_{m-n}^{-1} \sum_{k=m-n+1}^m \phi_k &= \phi_{m-n}^{-1} \sum_{k=m-n+1}^m \frac{1}{\|\mathcal{A}(k, m)Q_m x\|_k} \\ &\geq \phi_{m-n}^{-1} \sum_{k=m-n+1}^m \frac{1}{\|G\| \cdot \|Q_m x\|_m} \\ &\geq \frac{\phi_{m-n}^{-1}}{\|G\|^2 \|x\|_m} n. \end{aligned}$$

Hence,

$$\begin{aligned}
\|\mathcal{A}(m-2n, m)Q_mx\|_{m-2n} &= \|\mathcal{A}(m-2n, m-n)\mathcal{A}(m-n, m)Q_mx\|_{m-2n} \\
&\leq \|G\| \cdot \|\mathcal{A}(m-n, m)Q_mx\|_{m-n} \\
&= \|G\|\phi_{m-n}^{-1} \\
&\leq \frac{\|G\|^3\|x\|_m}{n}\phi_{m-n}^{-1} \sum_{k=m-n+1}^m \phi_k.
\end{aligned} \tag{31}$$

On the other hand,

$$\sum_{k=m-n+1}^m \mathcal{A}(m-n, k)Q_k\mathcal{A}(k, m)Q_mx\phi_k = \mathcal{A}(m-n, m)Q_mx \sum_{k=m-n+1}^m \phi_k,$$

which can be rewritten in the form

$$- \sum_{k=m-n+1}^m G_{m-n, k}\mathcal{A}(k, m)Q_mx\phi_k = \mathcal{A}(m-n, m)Q_mx \sum_{k=m-n+1}^m \phi_k. \tag{32}$$

For the sequence  $\mathbf{y}$  in (30), it follows from (12) that

$$(G\mathbf{y})_{m-n} = \sum_{k=m-n+1}^m G_{m-n, k}\mathcal{A}(k, m)Q_mx\phi_k.$$

Hence, by (32),

$$\begin{aligned}
\phi_{m-n}^{-1} \sum_{k=m-n+1}^m \phi_k &= \|(G\mathbf{y})_{m-n}\|_{m-n} \leq \|G\mathbf{y}\|_p \\
&\leq \|G\| \cdot \|\mathbf{y}\|_p = \|G\|n^{1/p}
\end{aligned}$$

and it follows from (31) that

$$\|\mathcal{A}(m-2n, m)P_mx\|_{m-2n} \leq \|G\|^4\|x\|_m n^{1/p-1}.$$

By (29) we obtain

$$\|\mathcal{A}(m-2n, m)Q_mx\|_{m-2n} \leq \frac{1}{e}\|x\|_m. \tag{33}$$

Now take  $k \leq l$  and write  $l-k$  in the form  $l-k = t \cdot 2n + r$ , with  $t \in \mathbb{N}$  and  $0 \leq r < 2n$ . It follows from (28) and (33) that

$$\begin{aligned}
\|\mathcal{A}(k, l)Q_lx\|_k &= \|\mathcal{A}(l-2nt-r, l)Q_lx\|_{l-2nt-r} \\
&\leq \frac{1}{e^t}\|\mathcal{A}(l-r, l)Q_lx\|_{l-r} \\
&\leq \frac{\|G\|}{e^t}\|Q_lx\|_l \\
&\leq 2\|G\|(1+\|G\|)e^{-(l-k)/(2n)}\|x\|_l.
\end{aligned}$$

We conclude that (27) holds taking  $D = 2\|G\|(1+\|G\|)$  and  $\lambda = 1/(2n)$ .  $\square$

This completes the proof of the theorem.  $\square$

## 4. ROBUSTNESS OF EXPONENTIAL DICHOTOMIES

In this section we show how the results of the former sections can be used to establish the persistence of the notion of an exponential dichotomy under sufficiently small linear perturbations. We shall use the characterization of exponential dichotomies via  $l^p$ -admissibility, although the same result can also be established using the characterization of exponential dichotomies via  $l^\infty$ -admissibility.

**Theorem 6.** *Let  $(A_m)_{m \in \mathbb{Z}}$  and  $(B_m)_{m \in \mathbb{Z}}$  be sequences of bounded linear operators on  $X$  such that:*

1.  $(A_m)_{m \in \mathbb{Z}}$  admits an exponential dichotomy with respect to a sequence of norms  $\|\cdot\|_m$ ;
2. there exists  $c > 0$  such that

$$\|(A_{n-1} - B_{n-1})x\|_n \leq c\|x\|_{n-1} \quad \text{for } n \in \mathbb{Z} \quad \text{and } x \in X. \quad (34)$$

If  $c$  is sufficiently small, then the sequence  $(B_m)_{m \in \mathbb{Z}}$  admits an exponential dichotomy with respect to the same sequence of norms.

*Proof.* Let  $Y$  be the Banach space introduced in Section 3 for  $1 \leq p < +\infty$  and let  $T$  be the linear operator defined by (10) in the domain  $\mathcal{D}(T)$  formed by all  $\mathbf{x} \in Y$  such that  $T\mathbf{x} \in Y$ . For  $\mathbf{x} \in \mathcal{D}(T)$  we consider the graph norm

$$\|\mathbf{x}\|'_p = \|\mathbf{x}\|_p + \|T\mathbf{x}\|_p.$$

Clearly, the operator

$$T: (\mathcal{D}(T), \|\cdot\|'_p) \rightarrow (Y, \|\cdot\|_p)$$

is bounded and for simplicity we denote it simply by  $T$ . Moreover, since  $T$  is closed,  $(\mathcal{D}(T), \|\cdot\|'_p)$  is a Banach space.

Now we define a linear operator  $L: \mathcal{D}(T) \rightarrow Y$  by

$$(L\mathbf{x})_n = x_n - B_{n-1}x_{n-1}, \quad \mathbf{x} = (x_n)_{n \in \mathbb{Z}}.$$

By (34) we have

$$\begin{aligned} \|(T - L)\mathbf{x}\|_p &= \left( \sum_{n=-\infty}^{\infty} \|(A_{n-1} - B_{n-1})x_{n-1}\|_n^p \right)^{1/p} \\ &\leq \left( \sum_{n=-\infty}^{\infty} c^p \|x_{n-1}\|_{n-1}^p \right)^{1/p} \\ &= c\|\mathbf{x}\|_p \leq c\|\mathbf{x}\|'_p \end{aligned} \quad (35)$$

for  $\mathbf{x} = (x_m)_{m \in \mathbb{Z}} \in Y$ . By Theorem 4, the operator  $T$  is invertible. Hence, it follows from (35) that if  $c$  is sufficiently small, then  $L$  is also invertible. By Theorem 5, we conclude that the sequence  $(B_m)_{m \in \mathbb{Z}}$  admits an exponential dichotomy with respect to the sequence of norms  $\|\cdot\|_m$ .  $\square$

## 5. NONUNIFORM EXPONENTIAL DICHOTOMIES

In this section we consider the notion of a nonuniform exponential dichotomy and we establish its connection with the notion of an exponential dichotomy with respect to a sequence of norms.

Let  $X = (X, \|\cdot\|)$  be a Banach space. We say that a sequence  $(A_m)_{m \in \mathbb{Z}}$  in  $B(X)$  admits a *nonuniform exponential dichotomy* if:

1. there exist projections  $P_m: X \rightarrow X$  for each  $m \in \mathbb{Z}$  satisfying

$$A_m P_m = P_{m+1} A_m \quad \text{for } m \in \mathbb{Z}$$

such that each map  $A_m|_{\ker P_m}: \ker P_m \rightarrow \ker P_{m+1}$  is invertible;

2. there exist constants  $\lambda, D > 0$  and  $\varepsilon \geq 0$  such that for each  $n, m \in \mathbb{Z}$  we have

$$\|\mathcal{A}(n, m)P_m\| \leq D e^{-\lambda(n-m) + \varepsilon|m|} \quad \text{for } n \geq m$$

and

$$\|\mathcal{A}(n, m)Q_m\| \leq D e^{-\lambda(m-n) + \varepsilon|m|} \quad \text{for } n \leq m,$$

where  $Q_m = \text{Id} - P_m$  and

$$\mathcal{A}(n, m) = (\mathcal{A}(m, n)|_{\ker P_n})^{-1}: \ker P_m \rightarrow \ker P_n$$

for  $n < m$ .

The following result shows that the notion of a nonuniform exponential dichotomy can be characterized in terms of the notion of an exponential dichotomy with respect to a sequence of norms.

**Proposition 7.** *The following properties are equivalent:*

1.  $(A_m)_{m \in \mathbb{Z}}$  admits a nonuniform exponential dichotomy;
2.  $(A_m)_{m \in \mathbb{Z}}$  admits an exponential dichotomy with respect to a sequence of norms  $\|\cdot\|_m$  satisfying

$$\|x\| \leq \|x\|_n \leq C e^{\varepsilon|n|} \|x\|, \quad n \in \mathbb{Z}, x \in X \quad (36)$$

for some constants  $C > 0$  and  $\varepsilon \geq 0$ .

*Proof.* Assume first that the sequence  $(A_m)_{m \in \mathbb{Z}}$  admits a nonuniform exponential dichotomy. For  $x \in X$  and  $n \in \mathbb{Z}$ , let

$$\|x\|_n = \sup_{m \geq n} (\|\mathcal{A}(m, n)P_n x\| e^{\lambda(m-n)}) + \sup_{m \leq n} (\|\mathcal{A}(m, n)Q_n x\| e^{\lambda(n-m)}).$$

It is easy to show that (36) holds. Moreover,

$$\begin{aligned} \|\mathcal{A}(m, n)P_n x\|_m &= \sup_{k \geq m} (\|\mathcal{A}(k, m)\mathcal{A}(m, n)P_n x\| e^{\lambda(k-m)}) \\ &= e^{-\lambda(m-n)} \sup_{k \geq m} (\|\mathcal{A}(k, n)P_n x\| e^{\lambda(k-n)}) \\ &\leq e^{-\lambda(m-n)} \|x\|_n \end{aligned}$$

for  $m \geq n$  and

$$\|\mathcal{A}(m, n)Q_n x\|_m \leq e^{-\lambda(n-m)} \|x\|_n$$

for  $m \leq n$ . This shows that  $(A_m)_m$  admits an exponential dichotomy with respect to the sequence of norms  $\|\cdot\|_m$ .

Conversely, assume that the sequence  $(A_m)_m$  admits an exponential dichotomy with respect to a sequence of norms  $\|\cdot\|_m$  satisfying (36) for some



constants  $C > 0$  and  $\varepsilon \geq 0$ . Then

$$\begin{aligned} \|\mathcal{A}(m, n)P_n x\| &\leq \|\mathcal{A}(m, n)P_n x\|_m \\ &\leq D e^{-\lambda(m-n)} \|x\|_n \\ &\leq C D e^{-\lambda(m-n) + \varepsilon|n|} \|x\| \end{aligned}$$

for  $x \in X$  and  $m \geq n$ . Similarly,

$$\|\mathcal{A}(m, n)Q_n x\| \leq C D e^{-\lambda(n-m) + \varepsilon|n|} \|x\|$$

for  $x \in X$  and  $m \leq n$ . Therefore,  $(A_m)_{m \in \mathbb{Z}}$  admits a nonuniform exponential dichotomy.  $\square$

The persistence of the notion of a nonuniform exponential dichotomy under sufficiently small linear perturbations now follows directly from Theorem 6 and Proposition 7.

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