ADMISSIBILITY, A GENERAL TYPE OF LIPSCHITZ SHADOWING AND STRUCTURAL STABILITY

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Abstract. For a general one-sided nonautonomous dynamics defined by a sequence of linear operators, we consider the notion of a uniform exponential dichotomy and we characterize it completely in terms of the admissibility of a large class of function spaces. We apply those results to show that structural stability of a diffeomorphism is equivalent to a very general type of Lipschitz shadowing property. Our results extend those in [37] in various directions.

1. Introduction

The notion of an exponential dichotomy, essentially introduced by Perron in [28], plays an important role in a large part of the theory of dynamical systems, such as, for example, in invariant manifold theory. We note that the theory of exponential dichotomies and its applications are very much developed. We refer to the books [8, 14, 15, 35] for details and further references. We particularly recommend [8] for a historical discussion. The reader may also consult the books [10, 11]. For the most recent developments we refer to [5].

Much of the work in the literature has been devoted to the study of the relationship between exponential dichotomies and the so-called admissibility property. The study of the admissibility property goes back to work of Perron [28] and referred originally to the existence of bounded solutions of the equation

$$x' = A(t)x + f(t)$$

in $\mathbb{R}^n$ for any bounded continuous perturbation $f\colon \mathbb{R}_0^+ \to \mathbb{R}^n$. For some of the most relevant early contributions in the area we refer to the work of Maizel [19], the books by Massera and Schäffer [22] (culminating the development initiated with their paper [21]) and by Dalec′kiĭ and Kreĭn [11]. Related results for discrete time were obtained by Coffman and Schäffer in [9]. We also refer to the book [18] for some early results in infinite-dimensional spaces. For a detailed list of references, we refer to the book by Chicone and Latushkin [8] (see in particular the final remarks of Chapters 3 and 4). We mention in particular the papers [23, 24, 33, 34] as an illustration of various approaches in the literature. For the most recent results we refer to [16, 1, 2, 3, 4]. Furthermore, the admissibility of certain pairs of spaces is

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related to the invertibility or the Fredholm properties of certain operators (see in particular [6, 7, 17, 25, 38] and the books [8, 10, 11, 15, 22]).

We note that most of the work in the literature consider the admissibility property with respect to the function spaces that are homogenous (roughly speaking, this means that the norm is invariant under translations). In [34], the author obtains a characterization of uniform exponential dichotomy in terms of the admissibility of general pairs of Banach sequence spaces (see subsection 2.2 for the definition) satisfying some additional properties. Recently, Todorov [37] established a discrete version of the theorem by Maizel [19] (and also of Pliss [32]) for the class of the function spaces that are not homogenous. More precisely, he showed that for the sequence $(A_n)_{n\geq 0}$ of invertible linear operators on $\mathbb{R}^d$, the following statements are equivalent:

1. the sequence $(A_n)_{n\geq 0}$ admits a uniform exponential dichotomy on $\mathbb{Z}^+$;
2. for each $\mathbf{y} = (y_n)_{n\geq 0} \in Y_w$ with $y_0 = 0$, there exists $\mathbf{x} = (x_n)_{n\geq 0} \in Y_{B,w}$ such that
$$x_{n+1} - A_n x_n = y_n, \quad \text{for each } n \geq 0.$$

Here, $w \geq 0$ is arbitrary and Y_w is the space of all sequences $\mathbf{x} = (x_n)_{n\geq 0} \subset \mathbb{R}^d$ such that
$$\|\mathbf{x}\| = \sup_{n\geq 0} \|x_n\|(1+n)^w < +\infty.$$
The main objective of the first part of our paper is to obtain a generalization of this result and of the corresponding version of theorem by Pliss. Using the notation of our Section 2, the results from [37] correspond to a very particular case when $B = l^\infty$ and $w_k = (1+k)^w$. We note that our approach for establishing exponential bounds along the stable and the unstable directions differs from the standard technique of substituting test sequences (see for example [15, 16, 34, 37]). Moreover, in contrast to the existing approaches, we are able to obtain bounds along the stable and unstable directions in a single step. Our methods are partially inspired by the characterization of hyperbolic sets presented in [12].

On the other hand, exponential dichotomies and admissibility concepts play also an important role in the shadowing theory (see [30, 31, 36, 37]). We emphasize that the shadowing theory has become a well developed and an important part of the general theory of dynamical systems. We refer to the books of Palmer [26] and Pilyugin [29] for a detailed exposition of the theory, further references and many historical comments. In the recent years, many of the papers have been devoted to the study of the relationship between structural stability and various type of shadowing properties. For example, while it was well-known that the structural stable diffeomorphisms have Lipschitz shadowing property (see [29]), it was only recently proved by Pilyugin and Tikhomirov [30] that the converse is also valid (see [27] for the related results for flows). We note that the proof of the main result from [30] uses theorems by Maizel and Pliss. More recently, Todorov [37] introduced a more general type of Lipschitz shadowing for diffeomorphisms and proved that it is equivalent to the structural stability. In a similar manner to that in [30] his proof uses a generalized version of the theorems by Maizel and Pliss established in [37] and described briefly in the previous paragraph.

In the second part of our paper we generalize further the results of [37]. More precisely, for a large class of function spaces B and sequences of positive numbers $\mathbf{w} = (w_k)_{k\geq 0}$, we introduce the notion of the $(B, \mathbf{w})$-Lipschitz shadowing for diffeomorphisms and prove that it is equivalent to the structural stability (see Section 4 for a detailed description of the results). Again, as in [30] and [37], the proof uses the appropriate version of the theorems by Maizel and Pliss established in the first part of our paper.

In particular, we show that every structurally stable diffeomorphism (and thus every Anosov diffeomorphism) has the $(B, \mathbf{w})$-Lipschitz shadowing property. Furthermore, one can easily modify the approach developed in [29] to show that this type of shadowing exists on a neighboorhood of each hyperbolic set. We refrain to formulate and prove this result explicitly since it would be a simple variation of already known results (just like the results in our subsection 4.3). We note that Pilyugin already showed that in a neighboorhood of each hyperbolic set there exists a weighted l^p-shadowing (see [29]). On the other hand, the relationship between structural stability and shadowing property (besides the already mentioned works) has also been discussed in [13] for the case of l^p-shadowing. Hence, the results in the second part of our paper can be seen as an attempt to both unify and generalize many of the known results in the literature.

2. Preliminaries

In this section we recall the basic notions of an exponential dichotomy and the admissible Banach sequence spaces. Furthermore, we introduce the class of Banach spaces that will play a central role in our paper. Those are defined in terms of an admissible Banach sequence space B and a sequence of positive numbers $\mathbf{w}$ whose terms will be called weights.

2.1. **Exponential dichotomy.** Let I be either $\mathbb{Z}$ or $\mathbb{Z}^+ = \{n \in \mathbb{Z} : n \geq 0\}$ or $\mathbb{Z}^- = \{n \in \mathbb{Z} : n \leq 0\}$. Furthermore, let $(A_m)_{m\in I}$ be the sequence of invertible linear operators on $\mathbb{R}^d$. We define the associated cocycle by

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1} \cdots A_n & \text{if } m > n, \\ \mathrm{Id} & \text{if } m = n, \\ A_m^{-1} \cdots A_{n-1}^{-1} & \text{if } m < n. \end{cases}$$

We say that the sequence $(A_m)_{m\in I}$ admits a *uniform exponential dichotomy* on I if:

1. there exist projections $P_m \colon \mathbb{R}^d \to \mathbb{R}^d$ for each $m \in I$ satisfying

$$\mathcal{A}(m,n)P_n = P_m\mathcal{A}(m,n) \quad \text{for} \quad m, n \in I; \tag{1}$$

2. there exist constants $\lambda, D > 0$ such that for every $n, m \in I$ we have

$$\|\mathcal{A}(m,n)P_n\| \leq De^{-\lambda(m-n)} \quad \text{for} \quad m \geq n \tag{2}$$

and

$$\|\mathcal{A}(m,n)Q_n\| \leq De^{-\lambda(n-m)} \quad \text{for} \quad m < n, \tag{3}$$

where $Q_n = \mathrm{Id} - P_n$.

2.2. **Banach sequence spaces.** In this subsection we present some basic definitions and properties from the theory of Banach sequence spaces.

Let $\mathcal{S}(I)$ be the set of all sequences $\mathbf{s} = (s_n)_{n\in I}$ of real numbers. We say that a linear subspace $B \subset \mathcal{S}(I)$ is a *normed sequence space* (over I) if there exists a norm $\|\cdot\|_B \colon B \to \mathbb{R}_0^+$ such that if $\mathbf{s}' \in B$ and $|s_n| \le |s'_n|$ for $n \in I$, then $\mathbf{s} \in B$ and $\|\mathbf{s}\|_B \le \|\mathbf{s}'\|_B$. If in addition $(B, \|\cdot\|_B)$ is complete, we say that B is a *Banach sequence space*.

Let B be a Banach sequence space. We say that B is *admissible* if:

1. $\chi_{\{n\}} \in B$ and $\|\chi_{\{n\}}\|_B > 0$ for $n \in I$, where χ_A denotes the characteristic function of the set $A \subset I$;
2. for each $\mathbf{s} = (s_n)_{n\in I} \in B$ and $m \in I$, the sequence $\mathbf{s}^m = (s_n^m)_{n\in I}$ defined by $s_n^m = s_{n+m}$ belongs to B and $\|\mathbf{s}^m\|_B = \|\mathbf{s}\|_B$ for $\mathbf{s} \in B$ and $m \in I$.

We present some examples of admissible Banach sequence spaces (over $\mathbb{Z}$).

Example 1. *The set* $l^\infty = \{\mathbf{s} \in \mathcal{S} : \sup_{n\in\mathbb{Z}}|s_n| < +\infty\}$ *is a Banach sequence space when equipped with the norm* $\|\mathbf{s}\| = \sup_{n\in\mathbb{Z}}|s_n|$.

Example 2. *For each* $p \in [1, \infty)$*, the set* $l^p = \{\mathbf{s} \in \mathcal{S} : \sum_{n\in\mathbb{Z}}|s_n|^p < +\infty\}$ *is a Banach sequence space when equipped with the norm* $\|\mathbf{s}\| = (\sum_{n\in\mathbb{Z}}|s_n|^p)^{1/p}$.

Example 3. *(Orlicz sequence spaces) Let* $\phi \colon (0, +\infty) \to (0, +\infty]$ *be a nondecreasing nonconstant left-continuous function. We set* $\psi(t) = \int_0^t \phi(s)\,ds$ *for* $t \ge 0$*. Moreover, for each* $\mathbf{s} \in \mathcal{S}$*, let* $M_\phi(\mathbf{s}) = \sum_{n\in\mathbb{Z}} \psi(|s_n|)$*. Then*

$$B = \{\mathbf{s} \in \mathcal{S} : M_\phi(c\mathbf{s}) < +\infty \text{ for some } c > 0\}$$

is a Banach sequence space when equipped with the norm

$$\|\mathbf{s}\| = \inf\{c > 0 : M_\phi(\mathbf{s}/c) \le 1\}.$$

The following auxiliary result is well-known (see [34] for example). We include the proof for the sake of completeness. Although we deal with the Banach sequence spaces over $\mathbb{Z}$, it is obvious that an analogous result can be formulated and proved for Banach sequence spaces over $\mathbb{Z}^+$ and $\mathbb{Z}^-$.

Proposition 1. *Let B be an admissible Banach sequence space over $\mathbb{Z}$.*

1. *If* $\mathbf{s}^1 = (s_n^1)_{n\in\mathbb{Z}}$ *and* $\mathbf{s}^2 = (s_n^2)_{n\in\mathbb{Z}}$ *are sequences in* $\mathcal{S}(\mathbb{Z})$ *and* $s_n^1 = s_n^2$ *for all but finitely many* $n \in \mathbb{Z}$*, then* $\mathbf{s}^1 \in B$ *if and only if* $\mathbf{s}^2 \in B$.
2. *If* $\mathbf{s}^n \to \mathbf{s}$ *in B when* $n \to \infty$*, then* $s_m^n \to s_m$ *when* $n \to \infty$*, for* $m \in \mathbb{Z}$.
3. *For each* $\mathbf{s} \in B$ *and* $\lambda \in (0, 1)$*, the sequences* $\mathbf{s}^1$ *and* $\mathbf{s}^2$ *defined by*

$$s_n^1 = \sum_{m\ge 0} \lambda^m s_{n-m} \quad \text{and} \quad s_n^2 = \sum_{m\ge 1} \lambda^m s_{n+m}$$

are in B, and

$$\|\mathbf{s}^1\|_B \le \frac{1}{1-\lambda}\|\mathbf{s}\|_B \quad \text{and} \quad \|\mathbf{s}^2\|_B \le \frac{\lambda}{1-\lambda}\|\mathbf{s}\|_B. \tag{4}$$

Proof. 1. Assume that $\mathbf{s}^1 \in B$ and let $I \subset \mathbb{Z}$ be the finite set of all integers $n \in \mathbb{Z}$ such that $s_n^1 \ne s_n^2$. We define $\mathbf{v} = (v_n)_{n\in\mathbb{Z}}$ by $v_n = 0$ if $n \notin I$ and $v_n = s_n^2 - s_n^1$ if $n \in I$. Since B is an admissible Banach sequence space, we have $\mathbf{v} \in B$ and thus $\mathbf{s}^2 = \mathbf{s}^1 + \mathbf{v} \in B$.

2. We have

$$|s^n_m - s_m|\chi_{\{m\}}(k) \le |s^n_k - s_k|$$

for $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. By the definition of a normed sequence space, we obtain

$$|s^n_m - s_m| \le \frac{1}{\|\chi_{\{0\}}\|_B}\|s^n - s\|_B$$

for $n \in \mathbb{Z}$ and the conclusion follows.

3. We define a sequence $\mathbf{v} = (v_n)_{n\in\mathbb{Z}}$ by $v_n = |s_n|$ for $n \in \mathbb{Z}$. Clearly, $\mathbf{v} \in B$ and $\|\mathbf{v}\|_B = \|\mathbf{s}\|_B$. Moreover,

$$\sum_{m\ge 0} \lambda^m \|\mathbf{v}^{-m}\|_B = \sum_{m\ge 0} \lambda^m \|\mathbf{v}\|_B = \frac{1}{1-\lambda}\|\mathbf{s}\|_B < +\infty.$$

Since B is complete, the series $\sum_{m\ge 0} \lambda^m \mathbf{v}^{-m}$ converges to some sequence $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in B$. It follows from the second property that

$$x_n = \sum_{m\ge 0} \lambda^m |s_{n-m}|$$

for $n \in \mathbb{Z}$. Since $|s^1_n| \le |x_n|$ for $n \in \mathbb{Z}$, we conclude that $\mathbf{s}^1 \in B$ and $\|\mathbf{s}^1\|_B \le \|\mathbf{x}\|_B$, which yields that the first inequality in (4) holds. One can show in a similar manner that $\mathbf{s}^2 \in B$ and that the second inequality in (4) holds. □

2.3. **Weights.** Throughout this paper $\mathbf{w} = (w_k)_{k\ge 0}$ will be a sequence of real numbers such that there exists $t > 0$ so that

$$w_k \ge t, \quad \text{for every } k \ge 0 \tag{5}$$

and with the property that for every $\lambda' > 0$, there exists $\lambda, L > 0$ such that:

$$e^{-\lambda'(m-n)}\frac{w_n}{w_m} \le Le^{-\lambda(m-n)} \quad \text{for every } m \ge n \ge 0, \tag{6}$$

and

$$e^{-\lambda'(n-m)}\frac{w_n}{w_m} \le Le^{-\lambda(n-m)} \quad \text{for every } n \ge m \ge 0. \tag{7}$$

It turns out that the second condition can be stated in a more transparent form.

Proposition 2. *The following statements are equivalent:*

1. *for every $\lambda' > 0$, there exists $\lambda, L > 0$ such that* (6) *and* (7) *hold;*
2. *for every $\varepsilon > 0$ there exists $C > 0$ such that*

$$\frac{w_n}{w_m} \le Ce^{\varepsilon|n-m|}, \quad m, n \ge 0. \tag{8}$$

Proof. Assume first that statement 1 holds. Take an arbitrary $\varepsilon > 0$ and let $\lambda' = \varepsilon$. It follows from the assumptions that there exist $L, \lambda > 0$ such that

$$e^{-\lambda'|m-n|}\frac{w_n}{w_m} \le Le^{-\lambda|m-n|} \le L,$$

for $m, n \ge 0$. Hence, (8) holds with $C = L$.

Assume now that the second statement holds and choose an arbitrary $\lambda' > 0$. Let $\varepsilon = \lambda'/2 > 0$. By our assumption, there exists $C > 0$ such that (8) holds. We conclude that (6) and (7) hold with $L = C$ and $\lambda = \lambda'/2$. □

The following proposition gives a large class of weights $\mathbf{w} = (w_k)_{k\geq 0}$ which satisfy the above properties. We remark that the weights in [37] correspond to a particular case when $p(k) = 1 + k$.

Proposition 3. *Assume that p is a polynomial with positive leading coefficient such that $p(k) > 0$ for $k \geq 0$. Given $w \geq 0$, we define*

$$w_k = p(k)^w, \quad k \geq 0.$$

Then, the sequence $\mathbf{w} = (w_k)_{k\geq 0}$ satisfies properties (5), (6) *and* (7).

Proof. Without loss of generality, we can assume that

$$0 < p(m) \leq p(m+1), \qquad \text{for every } m \in \mathbf{N}_0.$$

Obviously, (5) holds with $t = p(0)^w$. Take an arbitrary $\lambda' > 0$. For $m \geq n \geq 0$ we have $w_n \leq w_m$ and therefore (6) holds with $\lambda = \lambda'$ and $L = 1$. Assume now that $0 \leq m \leq n$. For $n \geq 2m$, we have

$$\begin{aligned} e^{-\lambda'(n-m)}\frac{w_n}{w_m} \leq e^{-\lambda'\frac{n}{2}}\frac{w_n}{w_m} \leq \frac{1}{w_1}e^{-\lambda'\frac{n}{2}}w_n &\leq \frac{1}{w_1}\left(\sup_{n\geq 0} e^{-\lambda'\frac{n}{4}}w_n\right)e^{-\lambda'\frac{n}{4}} \\ &\leq \frac{1}{w_1}\left(\sup_{n\geq 0} e^{-\lambda'\frac{n}{4}}w_n\right)e^{-\frac{\lambda'}{4}(n-m)} \end{aligned}$$

Since

$$\lim_{n\to\infty} e^{-\lambda'\frac{n}{4}}w_n = 0,$$

we conclude that (7) holds with $\lambda = \frac{\lambda'}{4}$ and some $L > 0$. Similarly, for $m \leq n \leq 2m$ we have

$$e^{-\lambda'(n-m)}\frac{w_n}{w_m} = e^{-\lambda'(n-m)}\frac{p(n)^w}{p(m)^w} \leq e^{-\lambda'(n-m)}\frac{p(2m)^w}{p(m)^w}.$$

We note that the sequence $(p(2m)^w/p(m)^w)_{m\in\mathbb{N}}$ converges, and therefore (7) holds with $\lambda = \lambda'$ and some $L > 0$. $\square$

2.4. **Important spaces.** In this subsection we introduce a class of Banach spaces that will play a crucial role in our paper.

Let B be an admissible Banach sequence space over I and let $\mathbf{w}$ be a sequence of weights. We define $Y_{B,\mathbf{w}} = Y_{B,\mathbf{w}}(I)$ to be the set of all $\mathbf{x} = (x_k)_{k\in I}$, $x_k \in \mathbb{R}^d$ with the property that the sequence $(w_{|k|}\|x_k\|)_{k\in I}$ belongs to B, where $\|\cdot\|$ denotes the standard Euclidean norm on $\mathbb{R}^d$.

Proposition 4. *$Y_{B,\mathbf{w}}$ is a Banach space with the norm*

$$\|\mathbf{x}\|_{B,\mathbf{w}} = \|(w_{|n|}\|x_n\|)_{n\in I}\|_B.$$

Proof. Let $(\mathbf{x}^k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $Y_{B,\mathbf{w}}$. Repeating arguments in the proof of Proposition 1, one can show that $(x_n^k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}^d$ for each $n \in I$. Hence, there exists

$$x_n = \lim_{k\to\infty} x_n^k, \quad \text{for every } n \in I.$$

For each $k \in \mathbb{N}$, let $\mathbf{s}^k = (w_{|n|}\|x_n^k\|)_{n\in I} \in B$. Since

$$\big|w_{|n|}\|x_n^k\| - w_{|n|}\|x_n^l\|\big| \leq w_{|n|}\|x_n^k - x_n^l\| \quad \text{for} \quad n \in I,$$

we conclude that

$$\|\mathbf{s}^k - \mathbf{s}^l\|_B \le \|\mathbf{x}^k - \mathbf{x}^l\|_{B,\mathbf{w}} \quad \text{for} \quad k, l \in \mathbb{N}.$$

Hence, $(\mathbf{s}^k)_{k\in\mathbb{N}}$ is a Cauchy sequence in B. Since B is complete, it follows from property 2 in Proposition 1 that $\mathbf{s}^k \to \mathbf{s}$ in B when $k \to \infty$, where $s_n = w_{|n|}\|x_n\|$ for $n \in I$. In particular, $\mathbf{x} = (x_n)_{n\in I} \in Y_{B,\mathbf{w}}$. One can easily verify that the sequence $(\mathbf{x}^k - \mathbf{x})_{k\in\mathbb{N}}$ converges to 0 in $Y_{B,\mathbf{w}}$, which implies that $(\mathbf{x}^k)_{k\in\mathbb{N}}$ converges to $\mathbf{x}$ in $Y_{B,\mathbf{w}}$. □

3. Admissibility and exponential dichotomies

In this section we establish generalizations of the theorems of Maizel [19] and Pliss [32] (which correspond to the special case when $B = l^\infty$ and $w_k = 1$), as well as their generalized versions established in [37] (which correspond to the special case when $B = l^\infty$ and $w_k = (1+k)^w$).

3.1. **Dichotomies on the positive half-line.**

In this subsection we obtain a characterization of exponential dichotomies on $\mathbb{Z}^+$. Let $(A_m)_{m\ge 0}$ be a sequence of invertible operators on $\mathbb{R}^d$. The following is our first result.

Theorem 5. *Assume that for each $y = (y_n)_{n\ge 0} \in Y_{B,\mathbf{w}}$ with $y_0 = 0$, there exists $\mathbf{x} = (x_n)_{n\ge 0} \in Y_{B,\mathbf{w}}$ such that*

$$x_{n+1} - A_n x_n = y_{n+1}, \quad \text{for every } n \ge 0. \tag{9}$$

Then, the sequence $(A_m)_{m\ge 0}$ admits an exponential dichotomy on $\mathbb{Z}^+$.

Proof. Set

$$X(0) = \{x \in \mathbb{R}^d : (\mathcal{A}(n,0)x)_{n\ge 0} \in Y_{B,\mathbf{w}}\}.$$

Clearly, $X(0)$ is a subspace of $\mathbb{R}^d$. Choose a subspace $Z \subset \mathbb{R}^d$ such that

$$\mathbb{R}^d = X(0) \oplus Z.$$

Lemma 1. *For each $\mathbf{y} = (y_n)_{n\ge 0} \in Y_{B,\mathbf{w}}$ with $y_0 = 0$, there exists a unique $\mathbf{x} = (x_n)_{n\ge 0} \in Y_{B,\mathbf{w}}$ with $x_0 \in Z$ such that (9) holds.*

Proof of the lemma. We first establish the existence of $\mathbf{x}$. Take $\mathbf{y} = (y_n)_{n\ge 0} \in Y_{B,\mathbf{w}}$ with $y_0 = 0$ and choose $\mathbf{x}^* = (x_n^*)_{n\ge 0} \in Y_{B,\mathbf{w}}$ such that

$$x_{n+1}^* - A_n x_n^* = y_{n+1}, \quad \text{for every } n \ge 0.$$

Write $x_0^* = z_1 + z_2$, $z_1 \in X(0)$, $z_2 \in Z$ and define

$$x_n = x_n^* - \mathcal{A}(n,0)z_1, \quad n \ge 0.$$

Then, $\mathbf{x} = (x_n)_{n\ge 0} \in Y_{B,\mathbf{w}}$, $x_0 \in Z$ and (9) holds.

In order to prove the uniqueness, it is sufficient to consider the situation when $\mathbf{y} = 0$. Assume that $\mathbf{x} = (x_n)_{n\ge 0} \in Y_{B,\mathbf{w}}$ with $x_0 \in Z$ satisfies $x_{n+1} = A_n x_n$ for $n \ge 0$. We have $x_n = \mathcal{A}(n,0)x_0$ for $n \ge 0$ and thus $x_0 \in X(0) \cap Z$. Hence, $x_0 = 0$ and $x_n = 0$ for every $n \ge 0$. We conclude that $\mathbf{x} = 0$ and the uniqueness is established. □

Let $Y_{B,\mathbf{w}}^0$ be the set of all $\mathbf{y} = (y_n)_{n\ge 0} \in Y_{B,\mathbf{w}}$ such that $y_0 = 0$. Clearly, $Y_{B,\mathbf{w}}^0$ is a closed subspace of $Y_{B,\mathbf{w}}$. We define a linear operator $T\colon \mathcal{D}(T) \subset Y_{B,\mathbf{w}} \to Y_{B,\mathbf{w}}^0$ by

$$(T\mathbf{x})_0 = 0 \quad \text{and} \quad (T\mathbf{x})_{n+1} = x_{n+1} - A_n x_n, \; n \ge 0$$

on the domain $\mathcal{D}(T)$ that consists of all $\mathbf{x} = (x_n)_{n\geq 0} \in Y_{B,\mathbf{w}}$, $x_0 \in Z$ such that $T\mathbf{x} \in Y^0_{B,\mathbf{w}}$.

Lemma 2. *$T\colon \mathcal{D}(T) \subset Y_{B,\mathbf{w}} \to Y^0_{B,\mathbf{w}}$ is a closed linear operator.*

Proof of the lemma. Let $(\mathbf{x}^k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{D}(T)$ converging to $\mathbf{x} \in Y_{B,\mathbf{w}}$ such that $T\mathbf{x}^k$ converges to $\mathbf{y} \in Y^0_{B,\mathbf{w}}$. It follows from the definition of Y_B and property 2 in Proposition 1 that

$$x_{n+1} - A_n x_n = \lim_{k\to\infty}(x^k_{n+1} - A_n x^k_n) = \lim_{k\to\infty}(T\mathbf{x}^k)_{n+1} = y_{n+1}$$

for $n \geq 0$, using the continuity of the linear operator A_n. Furthermore, $x_0 \in Z$. We conclude that $\mathbf{x} \in \mathcal{D}(T)$ and $T\mathbf{x} = \mathbf{y}$. This shows that the operator T is closed. □

For $\mathbf{x} \in \mathcal{D}(T)$ we consider the graph norm

$$\|\mathbf{x}\|'_{B,\mathbf{w}} = \|\mathbf{x}\|_{B,\mathbf{w}} + \|T\mathbf{x}\|_{B,\mathbf{w}}.$$

Clearly, the operator

$$T\colon (\mathcal{D}(T), \|\cdot\|'_{B,\mathbf{w}}) \to Y^0_{B,\mathbf{w}}$$

is bounded and from now on we denote it simply by T. It follows from Lemma 2 that $(\mathcal{D}(T), \|\cdot\|'_{B,\mathbf{w}})$ is a Banach space. By Lemma 1, T is an invertible operator.

For each $n \in \mathbb{N}$, we define $X(n)$ to be the set of all $x \in \mathbb{R}^d$ for which there exists a sequence $\mathbf{x} = (x_m)_{m\geq 0} \in Y_{B,\mathbf{w}}$ such that $x_m = \mathcal{A}(m,n)x$ for $m \geq n$. Moreover, let $Z(n) = \mathcal{A}(n,0)Z$, $n \geq 0$. Clearly, $X(n)$ and $Z(n)$ are subspaces of $\mathbb{R}^d$ for every $n \geq 0$.

Lemma 3. *We have*

$$\mathbb{R}^d = X(n) \oplus Z(n) \tag{10}$$

for every $n \geq 0$.

Proof of the lemma. For $n = 0$ there is nothing to prove. Take an arbitrary $n > 0$ and $v \in \mathbb{R}^d$. We define $\mathbf{y} = (y_m)_{m\geq 0}$ by $y_n = v$ and $y_m = 0$ for $m \neq n$. Clearly, $\mathbf{y} \in Y_{B,\mathbf{w}}$. By Lemma 1, there exists $\mathbf{x} = (x_m)_{m\geq 0} \in Y_{B,\mathbf{w}}$, $x_0 \in Z$ such that (9) holds. Hence,

$$x_n - A_{n-1}x_{n-1} = y_n = v \tag{11}$$

and

$$x_{m+1} = A_m x_m \quad \text{for } m \neq n-1. \tag{12}$$

It follows from (12) that $x_m = \mathcal{A}(m,n)x_n$ for $m \geq n$ and $A_{n-1}x_{n-1} = \mathcal{A}(n,0)x_0$. Therefore, $x_n \in X(n)$ and $A_{n-1}x_{n-1} \in Z(n)$ and by (11), $v \in X(n) + Z(n)$.

Take $v \in X(n) \cap Z(n)$ and choose $z \in Z$ such that $v = \mathcal{A}(n,0)z$. We define $\mathbf{x} = (x_m)_{m\geq 0}$ by $x_m = \mathcal{A}(m,0)z$, $m \geq 0$. It is easy to check that $\mathbf{x} \in \mathcal{D}(T)$ and $T\mathbf{x} = 0$. Hence, $\mathbf{x} = 0$ and $x_n = v = 0$. We conclude that (10) holds. □

Let $P_n\colon \mathbb{R}^d \to X(n)$ and $Q_n\colon \mathbb{R}^d \to Z(n)$ be the projections associated with the decomposition in (10). One can readily verify that (1) holds.

The following lemma will complete the proof of the theorem.

Lemma 4. *There exist constants $D, \lambda > 0$ such that* (2) *and* (3) *hold.*

Proof of the lemma. Fix $n > 0$ and $v \in \mathbb{R}^d$. Let $\mathbf{y}$ and $\mathbf{x}$ be as in the proof of Lemma 3. For each $r \geq 1$, we define a linear operator

$$B(r)\colon (\mathcal{D}(T), \|\cdot\|'_{B,\mathbf{w}}) \to Y^0_{B,\mathbf{w}}$$

by

$$(B(r)\boldsymbol{\nu})_0 = 0 \quad \text{and} \quad (B(r)\boldsymbol{\nu})_{m+1} = \begin{cases} r\nu_{m+1} - A_m\nu_m & \text{if } 0 \leq m < n, \\ \frac{1}{r}\nu_{m+1} - A_m\nu_m & \text{if } m \geq n. \end{cases}$$

We have $B(1) = T$ and

$$\|(B(r) - T)\boldsymbol{\nu}\|_{B,\mathbf{w}} \leq (r-1)\|\boldsymbol{\nu}\|'_{B,\mathbf{w}}$$

for $\boldsymbol{\nu} \in \mathcal{D}(T)$ and $r \geq 1$. In particular, this implies that $B(r)$ is invertible whenever $1 \leq r < 1 + 1/\|T^{-1}\|$, and

$$\|B(r)^{-1}\| \leq \frac{1}{\|T^{-1}\|^{-1} - (r-1)}.$$

Take $t = 1/r$ for a given $r \in (1, 1+1/\|T^{-1}\|)$ and let $\mathbf{z} \in \mathcal{D}(T)$ be the unique element such that $B(1/t)\mathbf{z} = \mathbf{y}$. Writing

$$D' = \frac{1}{\|T^{-1}\|^{-1} - (1/t - 1)},$$

we obtain

$$\begin{aligned} \|\mathbf{z}\|_{B,\mathbf{w}} \leq \|\mathbf{z}\|'_{B,\mathbf{w}} &= \|B(1/t)^{-1}\mathbf{y}\|'_{B,\mathbf{w}} \\ &\leq D'\|\mathbf{y}\|_{B,\mathbf{w}} = D'w_n\|\chi_{\{0\}}\|_B \cdot \|v\|. \end{aligned}$$

For each $m \geq 0$, let $x^*_m = t^{|m-n|-1}z_m$ and $\mathbf{x}^* = (x^*_m)_{m\geq 0}$. Obviously, $\mathbf{x}^* \in Y_{B,\mathbf{w}}$ and $x^*_0 \in Z$. Moreover, one can easily verify that $T\mathbf{x}^* = \mathbf{y}$ and therefore $\mathbf{x}^* = \mathbf{x}$. Thus,

$$\begin{aligned} \|x_m\| = \|x^*_m\| &= t^{|m-n|-1}\|z_m\| \\ &\leq \frac{1}{w_m\|\chi_{\{0\}}\|_B} t^{|m-n|-1}\|\mathbf{z}\|_{B,\mathbf{w}} \leq \frac{D'w_n}{tw_m} t^{|m-n|}\|v\| \end{aligned} \tag{13}$$

for $m \geq 0$. Moreover, it was shown in the proof of Lemma 3 that $P_n v = x_n$ and $Q_n v = -A_{n-1}x_{n-1}$. Hence, it follows from (6), (12) and (13) that there exists $\lambda, D > 0$ such that

$$\begin{aligned} \|\mathcal{A}(m,n)P_n v\| &= \|\mathcal{A}(m,n)x_n\| = \|x_m\| \\ &\leq De^{-\lambda(m-n)}\|v\| \end{aligned} \tag{14}$$

for $m \geq n$. This establishes (2) for $m \geq n > 0$. Furthermore, we have

$$\begin{aligned} \|\mathcal{A}(m,0)P_0 v\| = \|\mathcal{A}(m,1)P_1 A_0 v\| &\leq De^{-\lambda(m-1)}\|A_0 v\| \\ &\leq De^{\lambda}e^{-\lambda m}\|A_0\| \cdot \|v\| \end{aligned}$$

for $m > 0$ and $v \in \mathbb{R}^d$. This, together with the boundness of P_0 (to cover the case when $m = 0$) implies that (2) holds for all $m \geq n \geq 0$. Similarly, one can prove (3). □

□

The following is a converse of Theorem 5.

Theorem 6. *Assume that the sequence $(A_m)_{m\ge 0}$ admits an exponential dichotomy on $\mathbb{Z}^+$. Then, for each $y=(y_n)_{n\ge 0}\in Y_{B,\mathbf{w}}$ with $y_0=0$, there exists $\mathbf{x}=(x_n)_{n\ge 0}\in Y_{B,\mathbf{w}}$ such that* (9) *holds.*

Proof. Take $\mathbf{y}=(y_n)_{n\ge 0}\in Y_{B,\mathbf{w}}$ with $y_0=0$. For each $n\ge 0$, let

$$x_n^1=\sum_{m=0}^{n}\mathcal{A}(n,m)P_my_m$$

and

$$x_n^2=-\sum_{m=n+1}^{\infty}\mathcal{A}(n,m)Q_my_m.$$

It follows from (2) and (7) that there exist $\lambda', D'>0$ such that

$$w_n\|x_n^1\|\le\sum_{m=0}^{n}D\frac{w_n}{w_m}e^{-\lambda(n-m)}w_m\|y_m\|\le\sum_{m=0}^{n}D'e^{-\lambda'(n-m)}w_m\|y_m\|$$

for every $n\ge 0$. The last statement of Proposition 1 implies that

$$\bigg(\sum_{m=0}^{n}De^{-\lambda'(n-m)}w_m\|y_m\|\bigg)_{n\ge 0}\in B,$$

and consequently $(w_n\|x_n^1\|)_{n\ge 0}\in B$. We conclude that $(x_n^1)_{n\ge 0}\in Y_{B,\mathbf{w}}$. Similarly, $(x_n^2)_{n\ge 0}\in Y_{B,\mathbf{w}}$.

Now let $x_n=x_n^1+x_n^2$ for $n\ge 0$ and $\mathbf{x}=(x_n)_{n\ge 0}$. Obviously, $\mathbf{x}\in Y_{B,\mathbf{w}}$. Furthemore, it is easy to verify that (9) holds. ☐

3.2. **Dichotomies on the negative half-line.** In this subsection we state the versions of Theorems 5 and 6 for dichotomies on $\mathbb{Z}^-$. The proofs are quite similar to the proofs of Theorems 5 and 6 and thus we omit them. Let $(A_m)_{m\le 0}$ be the sequence of invertible linear operators on $\mathbb{R}^d$.

Theorem 7. *Assume that for each $y=(y_n)_{n\le 0}\in Y_{B,\mathbf{w}}$, there exists $\mathbf{x}=(x_n)_{n\le 0}\in Y_{B,\mathbf{w}}$ such that*

$$x_{n+1}-A_nx_n=y_{n+1},\quad \text{for every } n\le -1. \tag{15}$$

Then, the sequence $(A_m)_{m\le 0}$ admits an exponential dichotomy on $\mathbb{Z}^-$.

Now we state the converse of Theorem 7.

Theorem 8. *Assume that the sequence $(A_m)_{m\le 0}$ admits an exponential dichotomy on $\mathbb{Z}^-$. Then, for each $y=(y_n)_{n\le 0}\in Y_{B,\mathbf{w}}$ there exists $\mathbf{x}=(x_n)_{n\le 0}\in Y_{B,\mathbf{w}}$ such that* (15) *holds.*

3.3. **A generalization of a theorem by Pliss.** In this subsection we generalize the classical result of Pliss [32] as well as its generalization from [37]. We note that our arguments are much simpler then the ones in [37].

Let $(A_m)_{m\in\mathbb{Z}}$ be the sequence of invertible operators on $\mathbb{R}^d$. We define

$$\mathcal{S}=\{x\in\mathbb{R}^d:\lim_{k\to+\infty}\|\mathcal{A}(k,0)x\|=0\}$$

and

$$\mathcal{U}=\{x\in\mathbb{R}^d:\lim_{k\to-\infty}\|\mathcal{A}(k,0)x\|=0\}.$$

Clearly, $\mathcal{S}$ and $\mathcal{U}$ are subspaces of $\mathbb{R}^d$.

Theorem 9. *The following statements are equivalent:*

1. *for each* $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in Y_{B,\mathbf{w}}$*, there exists* $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_{B,\mathbf{w}}$ *such that*
$$x_{n+1} - A_n x_n = y_{n+1} \quad \textit{for } n \in \mathbb{Z}. \tag{16}$$
2. *the sequence* $(A_m)_{m\in\mathbb{Z}}$ *admits an exponential dichotomy on both half-lines and*
$$\mathbb{R}^d = \mathcal{S} + \mathcal{U}. \tag{17}$$

Proof. Assume first that statement 1 holds. We note that both the assumptions of Theorems 5 and 7 are satisfied. Hence, it follows that the sequence $(A_m)_{m\in\mathbb{Z}}$ admits an exponential dichotomy on $\mathbb{Z}^+$ and $\mathbb{Z}^-$. Take an arbitrary $v \in \mathbb{R}^d$ and define $\mathbf{y} = (y_n)_{n\in\mathbb{Z}}$ by $y_0 = v$. Clearly, $\mathbf{y} \in Y_{B,\mathbf{w}}$. Choose $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ such that (16) holds. Using the notation in the proofs of Theorems 5 and 7 we have that $x_0 \in X(0)$ and $-A_{-1}x_{-1} \in X'(0)$. Hence, x_0 belongs to a range of the projection P_0 associated with the dichotomy on $\mathbb{Z}^+$ and $A_{-1}x_{-1}$ belongs to a range of the projection Q_0 associated with the dichotomy on $\mathbb{Z}^-$. Now it follows directly from (2) and (3) that $x_0 \in \mathcal{S}$ and $A_{-1}x_{-1} \in \mathcal{U}$ and thus
$$v = x_0 - A_{-1}x_{-1} \in \mathcal{S} + \mathcal{U}.$$
We conclude that the statement 2 holds.

Assume now that statement 2 holds. Note that $\mathcal{S} = X(0)$ and $\mathcal{U} = X'(0)$. Take an arbitrary $\mathbf{y} = (y_n)_{n\in\mathbb{Z}}$ such that $y_n = 0$ for $n \le 0$. It follows from Theorem 6 that there exists $\mathbf{x}^* = (x_n^*)_{n\ge 0}$ such that
$$x_{n+1}^* - A_n x_n^* = y_{n+1}, \quad \text{for } n \ge 0.$$
Write $x_0^* = v_1 + v_2$, $v_1 \in X(0)$ and $v_2 \in X'(0)$. Furthermore, we define
$$x_n = \begin{cases} x_n^* - \mathcal{A}(n,0)v_1 & \text{if } n \ge 0; \\ \mathcal{A}(n,0)v_2 & \text{if } n < 0. \end{cases}$$
Then, $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_{B,\mathbf{w}}$ and (16) holds. Similarly, for $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in Y_{B,\mathbf{w}}$ such that $y_n = 0$ for $n > 0$, it follows from Theorem 8 that there exists $\mathbf{x}^* = (x_n^*)_{n\le 0}$ such that
$$x_{n+1}^* - A_n x_n^* = y_{n+1}, \quad \text{for } n \le -1.$$
Write $x_0^* = v_1 + v_2$, $v_1 \in X(0)$ and $v_2 \in X'(0)$. We define
$$x_n = \begin{cases} \mathcal{A}(n,0)v_1 & \text{if } n > 0; \\ x_n^* - \mathcal{A}(n,0)v_2 & \text{if } n \le 0. \end{cases}$$
Then, $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_{B,\mathbf{w}}$ and (16) holds. Finally, we note that every $\mathbf{y} \in Y_{B,\mathbf{w}}$ can be written in the form $\mathbf{y} = \mathbf{y}^1 + \mathbf{y}^2$ with $\mathbf{y}^1, \mathbf{y}^2 \in Y_{B,\mathbf{w}}$, $y_n^1 = 0$ for $n \le 0$ and $y_n^2 = 0$ for $n > 0$. We conclude that statement 1 holds. $\square$

We conclude this section by noting that one can generalize our results in several different directions. More precisely, one can establish the versions of all results for operators on an arbitrary Banach space. Furthermore, one can consider the general case of nonininvertible dynamics i.e. the case when the operators A_n are not necessarily invertible (this means that in the notion of exponential dichotomy we require only the invertibility along the unstable direction). Finally, one can establish the versions of Theorems 5 and 6 on the whole line $\mathbb{Z}$. We refrain to formulate and prove those results since will shall not need them in the rest of the paper.

4. Applications in shadowing theory

In this section we present some applications of our results in the shadowing theory. More precisely, we introduce a very general type of Lipschitz shadowing and prove that it is equivalent to structural stability.

4.1. **Preliminaries.** Let $\mathbf{w} = (w_k)_{k\geq 0}$ be a sequence of weights that satisfy properties (5), (6) and (7). Furthermore, let B be an admissible Banach sequence space that satisfies the following property: if $\mathbf{s} = (s_n)_{n\in\mathbb{Z}}$ is a sequence of real numbers such that there exists $Q > 0$ satisfying

$$\|\mathbf{s}\chi_{\{-N,\dots,0,\dots,N\}}\|_B \leq Q$$

for every $N \in \mathbb{N}$, then $\mathbf{s} \in B$ and $\|\mathbf{s}\|_B \leq Q$. We note that all examples of admissible Banach sequence spaces presented in the previous section satisfy this property.

Let M be a finite dimensional compact Riemannian manifold and let $f\colon M \to M$ be a C^1 diffeomorphism. A sequence $(x_k)_{k\in\mathbb{Z}}$, $x_k \in M$ is said to be a $(B, \mathbf{w})$ *d-pseudotrajectory* if

$$\|(w_{|n|}d(f(x_n), x_{n+1}))_{n\in\mathbb{Z}}\|_B \leq d.$$

Furthermore, we say that f has the $(B, \mathbf{w})$-Lipschitz shadowing property if there exist constants $L, d_0 > 0$ such that for every $d \leq d_0$ and every $(B, \mathbf{w})$ d-pseudotrajectory $(x_k)_{k\in\mathbb{Z}}$, there exists a point $p \in M$ such that

$$\|(w_{|n|}d(f^n(p), x_n))_{n\in\mathbb{Z}}\|_B \leq Ld.$$

We note that the classical Lipschitz shadowing property (see [29, 30]) corresponds to the case when $B = l^\infty$ and $w_k = 1$ and the Lipschitz shadowing studied in [37] corresponds to the case when $B = l^\infty$ and $w_k = (1 + k)^w$.

Finally, we recall that f is said to be structurally stable if there exists a neighborhood U of f in C^1 topology such that any diffeomorphism $g \in U$ is topologically conjugated to f. For $p \in M$, let T_pM denote the tangent space at a point p. We introduce two linear subspaces of T_pM:

$$\mathcal{S}(p) = \{v \in T_pM : \lim_{k\to+\infty} \|Df^k(p)\| = 0\}$$

and

$$\mathcal{U}(p) = \{v \in T_pM : \lim_{k\to-\infty} \|Df^k(p)\| = 0\}.$$

We will use the following result of Mañé [20].

Theorem 10. *A diffeomorphism f is structurally stable if and only if*

$$T_pM = \mathcal{S}(p) + \mathcal{U}(p),$$

for every $p \in M$.

4.2. **General type of shadowing implies structural stability.** In this subsection we prove that every diffeomorphism f with a $(B,\mathbf{w})$-Lipschitz shadowing property is structurally stable.

We first introduce some additional notation. For $N_1, N_2 \in \mathbb{Z}$, let $[N_1, N_2] = \{n \in \mathbb{Z} : N_1 \le n \le N_2\}$. Futhermore, if $(v_k)_{k\in[N_1,N_2]}$ is a finite sequence of real numbers then

$$\|(v_k)_{k\in[N_1,N_2]}\|_B := \|\mathbf{v}'\|_B,$$

where $\mathbf{v}' = (v'_n)_{n\in\mathbb{Z}} \in B$ is defined by $v'_n = v_n$ for $n \in [N_1, N_2]$ and $v'_n = 0$ otherwise. Finally, let

$$\alpha_B(n) = \|\chi_{\{0,1,\dots,n-1\}}\|_B, \quad n \in \mathbb{N}. \tag{18}$$

We need the following auxiliary result.

Lemma 5. *Assume that for a sequence $\mathbf{z} = (z_k)_{k\in\mathbb{Z}} \in Y_{B,\mathbf{w}}$, $\|\mathbf{z}\|_{B,\mathbf{w}} \le 1$ there exists $Q > 0$ such that for any $N \in \mathbb{N}$ there exists a sequence of vectors $(v_k^N)_{k\in[-N,N]}$ satisfying*

$$v_{k+1}^N - A_k v_k^N = z_{k+1}, \quad k \in [-N, N-1], \tag{19}$$

and $\|(w_{|k|}\|v_k^N\|)_{k\in[-N,N]}\|_B \le Q$. Then, there exists a sequence $\mathbf{v} = (v_k)_{k\in\mathbb{Z}} \in Y_{B,\mathbf{w}}$ such that

$$v_{k+1} - A_k v_k = z_{k+1} \quad \textit{for every } k \in \mathbb{Z}, \tag{20}$$

and $\|\mathbf{v}\|_{B,\mathbf{w}} \le Q$.

Proof. It is easy to show that for each $k \in \mathbb{Z}$ the sequence $(v_k^N)_{N\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}^d$. Hence, it has a convergent subsequence. Using a diagonal procedure, we can find a subsequence $(N_m)_{m\in\mathbb{N}}$ of $\mathbb{N}$ such that $(v_k^{N_m})_{m\in\mathbb{N}}$ converges for every $k \in \mathbb{Z}$. Let

$$v_k = \lim_{m\to\infty} v_k^{N_m}, \quad k \in \mathbb{Z}.$$

By passing to limits in (19), we conclude that the sequence $\mathbf{v} = (v_k)_{k\in\mathbb{Z}}$ satisfies (20). Take an arbitrary $M \in \mathbb{N}$. For every $m \ge M$, we have

$$\begin{aligned}\|\mathbf{v}\chi_{[-M,M]}\|_{B,\mathbf{w}} &\le \|(w_{|k|}\|v_k^{N_m}\|)_{k\in[-M,M]}\|_B \\ &\quad + \|(w_{|k|}\|v_k - v_k^{N_m}\|)_{k\in[-M,M]}\|_B \\ &\le Q + \|(w_{|k|}\|v_k - v_k^{N_m}\|)_{k\in[-M,M]}\|_B\end{aligned}$$

Letting $m \to \infty$, we obtain

$$\|(w_{|k|}\|v_k\|)_{k\in[-M,M]}\|_B = \|\mathbf{v}\chi_{[-M,M]}\|_{B,\mathbf{w}} \le Q.$$

Since M is arbitrary, we conclude that $\mathbf{v} \in Y_{B,\mathbf{w}}$ and $\|\mathbf{v}\|_{B,\mathbf{w}} \le Q$. □

Before proceeding, we observe that it follows from (6) and (7) that there exists $c > 0$ such that

$$\frac{w_{|k|}}{w_{|k+1|}} \le c \quad \text{and} \quad \frac{w_{|k+1|}}{w_{|k|}} \le c \quad \text{for every } k \in \mathbb{Z}. \tag{21}$$

The following is a main result of this subsection. Our approach is similar to that in [30] and [37].

Theorem 11. *Assume that $f\colon M \to M$ is a diffeomorphism with a $(B,\mathbf{w})$-Lipschitz shadowing property. Then, f is structurally stable.*

Proof. Fix a point $p \in M$ and let $A_k = Df(p_k)$, $p_k = f^k(p)$. Denote by $\mathcal{A}(m,n)$ the associated cocycle. It follows from Theorem 10 that in order to show that f is structurally stable, it is sufficient to prove that

$$T_pM = \mathcal{S} + \mathcal{U},$$

where

$$\mathcal{S} = \{v \in T_pM : \lim_{k\to+\infty} \|\mathcal{A}(k,0)v\| = 0\}$$

and

$$\mathcal{U} = \{v \in T_pM : \lim_{k\to-\infty} \|\mathcal{A}(k,0)v\| = 0\}.$$

To prove this we will use Theorem 9.

For $x \in M$ and $r > 0$, $B(r,x)$ will denote a ball in M with center in x and radius r. Similarly, $B_T(r,x)$ will denote a ball in T_xM with center in 0 and radius r. Let $\exp_x : T_xM \to M$ be a standard exponential mapping. Since M is compact, there exists $r > 0$ such that $\exp_x$ is a diffeomorphism of $B_T(r,x)$ onto its image and such that $\exp_x^{-1}$ is a diffeomorphism of $B(r,x)$ onto its image for every $x \in M$. Moreover, we can take r to be sufficiently small so that

$$d(\exp_x(v), \exp_x(w)) \le 2\|v-w\| \quad \text{for } v,w \in B_T(r,x), \tag{22}$$

and

$$\|\exp_x^{-1}(y) - \exp_x^{-1}(z)\| \le 2d(y,z) \quad \text{for } y,z \in B(r,x). \tag{23}$$

We consider mappings

$$F_k = \exp_{p_{k+1}}^{-1} \circ f \circ \exp_{p_k} : T_{p_k}M \to T_{p_{k+1}}M.$$

One can easily verify that

$$DF_k(0) = A_k.$$

Let L and d_0 be as in the notion of $(B,\mathbf{w})$-Lipschitz shadowing. In order to show that the first statement in Theorem 9 holds, we apply Lemma 5. Take $\mathbf{z} = (z_k)_{k\in\mathbb{Z}} \in Y_{B,\mathbf{w}}$ such that $\|\mathbf{z}\|_{B,\mathbf{w}} \le 1$ and fix $N \in \mathbb{N}$. Since M is compact, for every $\varepsilon > 0$ we can find $\delta > 0$ such that for every $\|v\| \le \delta$ we have

$$\|g_k(v)\| \le \frac{\varepsilon}{\alpha_B(2N+1)}\|v\|, \tag{24}$$

where $g_k(v) = F_k(v) - A_kv$. We define vectors

$$a_{-N} = 0, \quad a_{k+1} = A_ka_k + z_{k+1},\ k \in [-N, N-1].$$

Since the operators A_k are bounded, we can find a constant $C(N) > 0$ such that $\|a_k\| \le C(N)$ for every $k \in [-N, N-1]$. We also take $d > 0$ sufficiently small such that all points we consider in M belong to balls $B(r,p_k)$ and all the vectors in $T_{p_k}M$ that we consider belong to balls $B_T(r,p_k)$. Now we define a sequence of points $(\xi_k)_{k\in\mathbb{Z}}$ by $\xi_k = \exp_{p_k}(da_k)$ for $k \in [-N, N-1]$, $\xi_{k+N} = f^{k+1}(\xi_{N-1})$ for $k \ge 0$ and $\xi_{k-N} = f^k(\xi_{-N})$ for $k < 0$. Let $R = \max_{x\in M}\|Df(x)\|$. We may assume that ε satisfies

$$\varepsilon < \min\left\{w_{|k|}^{-1}/C(N), \frac{1}{2N}R^{-2N}w_{|k|}^{-1}\frac{1}{c'}\right\} \quad \text{for } k \in [-N, N-1], \tag{25}$$

where

$$c' = \frac{2}{t}\alpha_B(1)^{-1}2L(1+c) + 4C(N).$$

We now estimate $d(f(\xi_k), \xi_{k+1})$ for $k \in [-N, N-1]$. Since

$$\exp_{p_{k+1}}^{-1}(f(\xi_k)) = F_k(da_k) = A_k(da_k) + g_k(da_k)$$

and

$$\exp_{p_{k+1}}^{-1}(\xi_{k+1}) = da_{k+1} = d(A_k a_k + z_{k+1}),$$

we have (by (22)) that

$$\begin{aligned} d(f(\xi_k), \xi_{k+1}) &\leq 2\|F_k(da_k) - da_{k+1}\| \\ &= 2\|g_k(da_k) - dz_{k+1}\| \\ &\leq 2\|g_k(da_k)\| + 2d\|z_{k+1}\| \\ &\leq \frac{2\varepsilon d}{\alpha_B(2N+1)} \cdot C(N) + 2d\frac{w_{|k+1|}}{w_{|k+1|}}\|z_{k+1}\|. \end{aligned}$$

Hence, it follows from (21) and (25) that (we note that $\alpha_B(2N+1) \geq \alpha_B(2N)$)

$$w_{|k|} d(f(\xi_k), \xi_{k+1}) \leq \frac{2d}{\alpha_B(2N)} + 2cdw_{|k+1|}\|z_{k+1}\|$$

for $k \in [-N, N-1]$. Noting that $d(f(\xi_k), \xi_{k+1}) = 0$ for $k \notin [-N, N-1]$, we conclude that $(\xi_k)_{k\in\mathbb{Z}}$ is a $(B, \mathbf{w})$ $(2d + 2cd)$-pseudotrajectory. We may assume that d is sufficiently small so that $2d + 2cd < d_0$. Since f has the $(B, \mathbf{w})$-Lipschitz shadowing property, there exists a trajectory $(y_k)_{k\in\mathbb{Z}}$ such that

$$\|(w_{|k|} d(\xi_k, y_k))_{k\in\mathbb{Z}}\|_B \leq 2L(1+c)d \tag{26}$$

It follows that

$$d(\xi_k, y_k) \leq w_{|k|}^{-1} \alpha_B(1)^{-1} 2L(1+c)d, \quad \text{for every } k \in \mathbb{Z}. \tag{27}$$

Let $t_k = \exp_{p_k}^{-1}(y_k)$. Then,

$$t_{k+1} = F_k(t_k) = A_k t_k + g_k(t_k).$$

By (22), (23) and (27) we have

$$\begin{aligned} \|t_k\| \leq 2d(y_k, p_k) &\leq 2d(y_k, \xi_k) + 2d(\xi_k, p_k) \\ &\leq 2w_{|k|}^{-1} \alpha_B(1)^{-1} 2L(1+c)d + 4d\|a_k\| \\ &\leq \left(\frac{2}{t}\alpha_B(1)^{-1} 2L(1+c) + 4C(N)\right)d \\ &= c'd. \end{aligned}$$

Now we set

$$b_k = \mathcal{A}(k, -N)t_{-N} \quad \text{and} \quad c_k = t_k - b_k,\ k \in [-N, N].$$

Then,

$$c_{-N} = 0 \quad \text{and} \quad c_{k+1} = A_k c_k + g_k(t_k),\ k \in [-N, N-1].$$

Hence,

$$c_k = \sum_{j=0}^{k+N-1} \mathcal{A}(k, k-j) g_{k-j-1}(t_{k-j-1}).$$

It follows from (24) that

$$\|c_k\| \leq 2N \cdot R^{2N} \cdot \frac{\varepsilon}{\alpha_B(2N+1)} \cdot c'd, \quad k \in [-N, N]. \tag{28}$$

By (25),

$$w_{|k|}\|c_k\| \le \frac{d}{\alpha_B(2N+1)}, \quad k \in [-N, N]. \tag{29}$$

We define

$$v_k = a_k - \frac{b_k}{d}, \quad k \in [-N, N].$$

One can easily verify that

$$v_{k+1} - A_k v_k = z_{k+1}, \quad k \in [-N, N-1].$$

Furthermore, we have

$$\begin{aligned}\|a_k - \frac{b_k}{d}\| = \frac{1}{d}\|da_k - b_k\| &= \frac{1}{d}\|(\exp_{p_k}^{-1}(\xi_k) - \exp_{p_k}^{-1}(y_k)) + c_k\| \\ &\le \frac{1}{d}\|\exp_{p_k}^{-1}(\xi_k) - \exp_{p_k}^{-1}(y_k)\| + \frac{1}{d}\|c_k\| \\ &\le \frac{2}{d}d(\xi_k, y_k) + \frac{1}{d}\|c_k\|\end{aligned}$$

and therefore

$$w_{|k|}\|v_k\| = w_{|k|}\|a_k - \frac{b_k}{d}\| \le \frac{2}{d}w_{|k|}d(\xi_k, y_k) + \frac{w_{|k|}}{d}\|c_k\|$$

for every $k \in [-N, N]$. It follows from (26) and (29) that

$$\|(w_{|k|}\|v_k\|)_{k\in[-N,N]}\|_B \le 4L(1+c) + 1.$$

We conclude that the assumption of Lemma 5 holds and hence the first statement in Theorem 9 is valid. By Theorem 10, f is structurally stable and the proof is complete. □

4.3. **Structural stability implies general type of shadowing.** In this subsection we obtain the converse of Theorem 11. The idea is to modify the approach developed in [29], where it is proved that structural stability implies classical Lipschitz shadowing.

We continue to consider sequences of weights $\mathbf{w}$ which satisfy (5), (6) and (7). Also, let c be as in (21). We consider a family $\{H_k\}_{k\in\mathbb{Z}}$ of Banach spaces. For an admissible Banach sequence space B, we denote by $Y_{B,\mathbf{w}}$ the space of all sequences $\mathbf{v} = (v_n)_{n\in\mathbb{Z}}$, $v_n \in H_n$ such that

$$\|\mathbf{v}\|_{B,\mathbf{w}} = \|(w_{|n|}\|v_n\|)_{n\in\mathbb{Z}}\|_B < +\infty.$$

Then, $(Y_{B,\mathbf{w}}, \|\cdot\|_{B,\mathbf{w}})$ is a Banach space.

The proof of the following result can be obtained by repeating the proof of Lemma 1.3.1. from [29].

Lemma 6. *Let $(\phi_k)_k$ be a sequence of maps $\phi_k\colon H_k \to H_{k+1}$ of the form*

$$\phi_k(v) = A_k v + \psi_{k+1}(v),$$

where A_k are linear maps. Assume that for numbers $N_0, \kappa, \Delta > 0$:

1. *there exists a linear operator $\mathcal{G}\colon Y_{B,\mathbf{w}} \to Y_{B,\mathbf{w}}$ such that*
   (a) $\|\mathcal{G}\| \le N_0$;
   (b) *if $\mathbf{z} = (z_k)_{k\in\mathbb{Z}} \in Y_{B,\mathbf{w}}$, then the sequence $\mathbf{u} = (u_k)_{k\in\mathbb{Z}}$ defined by $\mathbf{u} = \mathcal{G}\mathbf{z}$ satisfies*

$$u_{k+1} = A_k u_k + z_{k+1}, \quad \text{for every } k \in \mathbb{Z};$$

2. *we have*
$$\|\psi_{k+1}(v)-\psi_{k+1}(v')\| \le \frac{\kappa}{c}\|v-v'\| \quad \text{for } \|v\|, \|v'\| \le \frac{\Delta}{t\alpha_B(1)}, \tag{30}$$
where t is as in (5) *and $\alpha_B(1)$ as in* (18) *for $n=1$;*
3. $\kappa N_0 < 1$.

Set
$$L = \frac{N_0}{1-\kappa N_0} \quad \text{and} \quad d_0 = \frac{\Delta}{L}.$$
If
$$\|(\phi_k(0))_{k\in\mathbb{Z}}\|_{B,\mathbf{w}} \le d \le d_0,$$
then there exist $v_k \in H_k$ such that $\phi_k(v_k) = v_{k+1}$ and
$$\|(v_k)_{k\in\mathbb{Z}}\|_{B,\mathbf{w}} \le Ld.$$

The following auxiliary result is a direct consequence of (6) and (7).

Lemma 7. *For every $\lambda' > 0$ there exists $\lambda, L > 0$ such that for every $n, m \in \mathbb{Z}$:*

1. $$e^{-\lambda'(m-n)}\frac{w_{|n|}}{w_{|m|}} \le Le^{-\lambda(m-n)}, \quad \text{for } m \ge n; \tag{31}$$
2. $$e^{-\lambda'(n-m)}\frac{w_{|n|}}{w_{|m|}} \le Le^{-\lambda(n-m)}, \quad \text{for } m \le n. \tag{32}$$

Proof. We will prove (31). The proof of (32) is completely analogous. Take $m \ge n$. We distinguish three cases. If $m \ge n \ge 0$, then (31) reduces to (6) and there is nothing to prove. If $0 \ge m \ge n$, then it follows from (7) that for a given $\lambda' > 0$ there exists $L, \lambda > 0$ (independent on n and m) such that
$$e^{-\lambda'(|n|-|m|)}\frac{w_{|n|}}{w_{|m|}} \le Le^{-\lambda(|n|-|m|)}.$$
Obviously, the above inequality is equivalent to (31). Finally, we consider the case when $m \ge 0 \ge n$. We can write
$$e^{-\lambda'(m-n)}\frac{w_{|n|}}{w_{|m|}} = e^{-\lambda' m}\frac{w_0}{w_m}\cdot e^{-\lambda'|n|}\frac{w_{|n|}}{w_0}.$$
By previous cases, there exists $L, \lambda > 0$ (independent on m and n) such that
$$e^{-\lambda' m}\frac{w_0}{w_m} \le Le^{-\lambda m} \quad \text{and} \quad e^{-\lambda'|n|}\frac{w_{|n|}}{w_0} \le Le^{\lambda n},$$
and consequently
$$e^{-\lambda'(m-n)}\frac{w_{|n|}}{w_{|m|}} \le L^2 e^{-\lambda(m-n)}.$$
□

Theorem 12. *Assume that*

1. *there exist $\lambda' \in (0,1)$, $N \ge 1$, and projections $P_k, Q_k \colon H_k \to H_k$ with ranges S_k and U_k respectively such that:*
   (a) $\|P_k\|, \|Q_k\| \le N$, $P_k + Q_k = \mathrm{Id}$;
   (b) $\|A_k|S_k\| \le \lambda'$, $A_k S_k \subset S_{k+1}$;

2. *if* $U_{k+1} \neq \{0\}$, *then there exist linear mappings* $B_k \colon U_{k+1} \to H_k$ *such that*
$$B_k U_{k+1} \subseteq U_k, \ \|B_k\| \le \lambda', A_k B_k = \mathrm{Id}; \tag{33}$$
3. *there exist* $\kappa, \Delta > 0$ *such that* (30) *holds and*
$$\kappa N_1 < 1,$$
where
$$N_1 = NL' \frac{1+\lambda}{1-\lambda},$$
with respect to some $\lambda \in (0,1)$ *and* $L' > 0$ *that depend only on weights and* λ'.

Set
$$L = \frac{N_1}{1-\kappa N_1} \quad \text{and} \quad d_0 = \frac{\Delta}{L}.$$
If
$$\|(\phi_k(0))_{k\in\mathbb{Z}}\|_{B,\mathbf{w}} \le d \le d_0,$$
then there exist $v_k \in H_k$ *such that* $\phi_k(v_k) = v_{k+1}$ *and*
$$\|(v_k)_{k\in\mathbb{Z}}\|_{B,\mathbf{w}} \le Ld.$$

Proof. We define $\mathcal{G} \colon Y_{B,\mathbf{w}} \to Y_{B,\mathbf{w}}$ by
$$(\mathcal{G}\mathbf{z})_n = \sum_{m=-\infty}^{n} \mathcal{A}(n,m) P_m z_m - \sum_{m=n+1}^{\infty} \mathcal{B}(n,m) Q_m z_m,$$
where $\mathcal{A}$ and $\mathcal{B}$ are cocycles associated to $(A_k)_k$ and $(B_k)_k$ respectively. We prove that $\mathcal{G}$ is well-defined and bounded. Indeed, we have
$$\|(\mathcal{G}\mathbf{z})_n\| \le \sum_{m=-\infty}^{n} N(\lambda')^{n-m}\|z_m\| + \sum_{m=n+1}^{\infty} N(\lambda')^{m-n}\|z_m\|,$$
and consequently
$$\begin{aligned} w_{|n|}\|(\mathcal{G}\mathbf{z})_n\| &\le N \sum_{m=-\infty}^{n} (\lambda')^{n-m} \frac{w_{|n|}}{w_{|m|}} w_{|m|}\|z_m\| \\ &\quad + N \sum_{m=n+1}^{\infty} (\lambda')^{m-n} \frac{w_{|n|}}{w_{|m|}} w_{|m|}\|z_m\|. \end{aligned}$$
It follows from Lemma 7 that there exist $\lambda \in (0,1)$ and $L' > 0$ such that
$$w_{|n|}\|(\mathcal{G}\mathbf{z})_n\| \le NL' \sum_{m=-\infty}^{n} \lambda^{n-m} w_{|m|}\|z_m\| + NL' \sum_{m=n+1}^{\infty} \lambda^{m-n} w_{|m|}\|z_m\|.$$
Proceeding as in the proof of Theorem 6 (and using (4)) we obtain
$$\|\mathcal{G}\| \le N_1.$$
Moreover, one can easily show that $\mathcal{G}$ satisfies property (b) in the statement of Lemma 6. The conclusion of the theorem now follows directly from Lemma 6. □

Theorem 13. *Let* $f \colon M \to M$ *be a structurally stable diffeomorphism. Then,* f *has the* $(B, \mathbf{w})$*-Lipschitz shadowing property.*

Proof. The proof goes along the lines of the proof of Theorem 2.2.7 from [29], using Theorem 12 instead of Theorem 1.3.1. from [29] when necessary. □

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