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FROM ONE-SIDED DICHOTOMIES TO TWO-SIDED DICHOTOMIES

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Abstract. For a general nonautonomous dynamics on a Banach space, we give a necessary and sufficient condition so that the existence of one-sided exponential dichotomies on the past and on the future gives rise to a two-sided exponential dichotomy. The condition is that the stable space of the future at the origin and the unstable space of the past at the origin generate the whole space. We consider the general cases of a noninvertible dynamics as well as of a nonuniform exponential dichotomy and a strong nonuniform exponential dichotomy (for the latter, besides the requirements for a nonuniform exponential dichotomy we need to have a minimal contraction and a maximal expansion). Both notions are ubiquitous in ergodic theory. Our approach consists in reducing the study of the dynamics to one with uniform exponential behavior with respect to a family of norms and then using the characterization of uniform hyperbolicity in terms of an admissibility property in order to show that the dynamics admits a two-sided exponential dichotomy. As an application, we give a complete characterization of the set of Lyapunov exponents of a Lyapunov regular dynamics, in an analogous manner to that in the Sacker–Sell theory.

1. **Introduction.** For a linear nonautonomous dynamics on a Banach space, we give a necessary and sufficient condition so that the existence of one-sided nonuniform exponential dichotomies on the past and on the future gives rise to a two-sided nonuniform exponential dichotomy. More precisely, we consider a linear nonautonomous dynamics with discrete time defined by a sequence A_n of linear operators or a nonautonomous dynamics with continuous time given by an evolution family say determined by a nonautonomous linear equation $x' = A(t)x$ on a Banach space. For example, in the case of discrete time we give a necessary and sufficient condition so that a sequence of linear operators admitting nonuniform exponential

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dichotomies both on $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^-$ also admits a nonuniform exponential dichotomy on $\mathbb{Z}$. The condition is that the stable space of $\mathbb{Z}_0^+$ at the origin and the unstable space of $\mathbb{Z}_0^-$ at the origin generate the whole space. We obtain an analogous result in the case of continuous time although it requires a separate approach. Moreover, we consider the general case of a noninvertible dynamics.

A principal motivation for the notion of a nonuniform exponential dichotomy is that it occurs naturally in measure-preserving dynamics. Namely, let $f\colon M \to M$ be a diffeomorphism and let μ be an f-invariant finite measure on M. If $\log^+\|df\|$ is μ-integrable, then for μ-almost every $x \in M$ if

$$\limsup_{n\to\infty} \frac{1}{n} \log\|d_x f^n v\| \neq 0$$

for all $v \neq 0$, then the sequence $A_n(x) = d_{f^n(x)} f$ admits a nonuniform exponential dichotomy (see for example [3]). In fact, the sequence even admits a strong nonuniform exponential dichotomy (this means that besides the requirements for a nonuniform exponential dichotomy one also assumes that there is a minimal contraction and a maximal expansion). Thus, from the point of view of ergodic theory the nonuniform exponential behavior is ubiquitous. We refer the reader to [3, 8] for details and references.

Our approach consists in reducing the study of the dynamics to one with uniform exponential behavior with respect to a family of norms and then using the characterization of uniform hyperbolicity in terms of an admissibility property (partly inspired by related approaches in [15, 17]) in order to show that the dynamics admits a two-sided exponential dichotomy. In the particular case of uniform exponential dichotomies our results are closely related to work of Pliss in [31] (see the discussion after Theorem 2.9).

While it is difficult to indicate an original reference for considering families of norms in the classical uniform theory (both for discrete and continuous time), in the nonuniform theory it first occurred in Pesin's work on nonuniform hyperbolicity and smooth ergodic theory [28, 29] (see also the detailed description in [4]). Our notion of an exponential dichotomy with respect to a family of norms is motivated by his approach (see [8] for a detailed discussion), although now having in mind the characterization of the hyperbolicity in terms of an admissibility.

On the other hand, the study of admissibility goes back to pioneering work of Perron in [27] and referred originally to the existence of bounded solutions of the equation

$$x' = A(t)x + f(t)$$

in $\mathbb{R}^n$ for any bounded continuous perturbation $f\colon \mathbb{R}_0^+ \to \mathbb{R}^n$. It turns out that this property is related to the conditional stability of the linear equation $x' = A(t)x$. A corresponding study for discrete time was initiated by Li in [21]. It was proved by Maizel' in [22] (for an integrally bounded coefficient matrix) and by Coppel in [12] (in the general case) that the admissibility property on $\mathbb{R}_0^+$ implies that the linear equation admits an exponential dichotomy. For the description of some of the most relevant early contributions in the area see the books by Massera and Schäffer [24] (see also [23]), by Dalec′kiĭ and Kreĭn [14] and by Coppel [13]. Related results for discrete time were obtained by Coffman and Schäffer in [11]. See also [20] for the description of some early results in infinite-dimensional spaces. For a detailed list of references, we refer the reader to [10] and for more recent work to Huy [16] and Todorov [33].

Finally, as an application, we give a complete characterization of the set of Lyapunov exponents of a Lyapunov regular dynamics in an analogous manner to that in the Sacker–Sell theory (see [32]), although now expressed in terms of nonuniform exponential dichotomies with an arbitrarily small nonuniform part.

2. **Discrete time.**

In this section we give a necessary and sufficient condition so that a sequence of linear operators admitting nonuniform exponential dichotomies both on $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^-$ also admits a nonuniform exponential dichotomy on $\mathbb{Z}$. The condition is that the stable space of $\mathbb{Z}_0^+$ at the origin and the unstable space of $\mathbb{Z}_0^-$ at the origin generate the whole space. We also consider the case of strong nonuniform exponential dichotomies.

2.1. **Nonuniform exponential dichotomies.**

Let $X = (X, \|\cdot\|)$ be a Banach space and let $B(X)$ be the set of all bounded linear operators on X. Moreover, let $I \in \{\mathbb{Z}_0^+, \mathbb{Z}_0^-, \mathbb{Z}\}$ be an interval, where

$$\mathbb{Z}_0^+ = \{n \in \mathbb{Z} : n \geq 0\} \quad \text{and} \quad \mathbb{Z}_0^- = \{n \in \mathbb{Z} : n \leq 0\}.$$

Given a sequence $(A_m)_{m\in I}$ in $B(X)$, we define

$$\mathcal{A}(n,m) = \begin{cases} A_{n-1}\cdots A_m & \text{if } n > m, \\ \mathrm{Id} & \text{if } n = m \end{cases} \tag{1}$$

for $n, m \in I$ with $n \geq m$. We say that $(A_m)_{m\in I}$ admits a *nonuniform exponential dichotomy on I* if:

1. there exist projections $P_m \in B(X)$ for $m \in I$ satisfying
$$\mathcal{A}(n,m)P_m = P_n\mathcal{A}(n,m) \quad \text{for} \quad n \geq m \tag{2}$$
such that each map
$$\mathcal{A}(n,m)|\operatorname{Ker} P_m \colon \operatorname{Ker} P_m \to \operatorname{Ker} P_n$$
is invertible;
2. there exist constants $\lambda, D > 0$ and $\varepsilon \geq 0$ such that for $n, m \in I$ we have
$$\|\mathcal{A}(n,m)P_m\| \leq De^{-\lambda(n-m)+\varepsilon|m|} \quad \text{for} \quad n \geq m \tag{3}$$
and
$$\|\mathcal{A}(n,m)Q_m\| \leq De^{-\lambda(m-n)+\varepsilon|m|} \quad \text{for} \quad n \leq m, \tag{4}$$
where $Q_m = \mathrm{Id} - P_m$ and
$$\mathcal{A}(n,m) = (\mathcal{A}(m,n)|\operatorname{Ker} P_n)^{-1} \colon \operatorname{Ker} P_m \to \operatorname{Ker} P_n$$
for $n < m$.

More generally, given a sequence of norms $\|\cdot\|_m$ for $m \in I$ on X, we say that $(A_m)_{m\in I}$ admits a *nonuniform exponential dichotomy on I with respect to the sequence of norms* $\|\cdot\|_m$ if conditions 1–2 hold with (3) and (4) replaced respectively by

$$\|\mathcal{A}(n,m)P_m x\|_n \leq De^{-\lambda(n-m)+\varepsilon|m|}\|x\|_m \quad \text{for} \quad n \geq m,\ x \in X \tag{5}$$

and

$$\|\mathcal{A}(n,m)Q_m x\|_n \leq De^{-\lambda(m-n)+\varepsilon|m|}\|x\|_m \quad \text{for} \quad n \leq m,\ x \in X. \tag{6}$$

Example 2.1. Given $\omega < 0$ and $\varepsilon, c > 0$ such that $\omega + \varepsilon < 0$, consider the real numbers

$$A_m = \begin{cases} e^{\omega+\varepsilon[(-1)^m m-1/2]-c(2m+1)} & \text{if } m \geq 0, \\ e^{-\omega+\varepsilon[(-1)^{m+1} m-1/2]-c(2m+1)} & \text{if } m < 0. \end{cases}$$

For $n \geq m \geq 0$ we have

$$\begin{aligned} \mathcal{A}(n,m) &= e^{(\omega-\varepsilon/2)(n-m)+\varepsilon\sum_{k=m}^{n-1}(-1)^k k - c(n^2-m^2)} \\ &= e^{(\omega-\varepsilon/2)(n-m)+\varepsilon(-1)^{n-1}\lfloor n/2\rfloor-\varepsilon(-1)^{m-1}\lfloor m/2\rfloor-c(n^2-m^2)} \\ &\leq e^{\omega(n-m)+\varepsilon m} \end{aligned} \tag{7}$$

(see [9]) and thus $(A_m)_{m\geq 0}$ admits a nonuniform exponential dichotomy on $\mathbb{Z}_0^+$ with $P_m = \mathrm{Id}$ for $m \geq 0$. On the other hand, for $n \leq m \leq 0$ we have

$$\begin{aligned} \mathcal{A}(m,n)^{-1} &= e^{(\omega+\varepsilon/2)(m-n)+\varepsilon\sum_{k=n}^{m-1}(-1)^k k - c(n^2-m^2)} \\ &= e^{(\omega+\varepsilon/2)(m-n)+\varepsilon(-1)^{|n|+1}\lfloor(|n|+1)/2\rfloor-\varepsilon(-1)^{|m|+1}\lfloor(|m|+1)/2\rfloor-c(n^2-m^2)} \\ &\leq e^{\varepsilon+(\omega+\varepsilon)(m-n)+\varepsilon|m|} \end{aligned} \tag{8}$$

(see [9]) and thus $(A_m)_{m\leq 0}$ admits a nonuniform exponential dichotomy on $\mathbb{Z}_0^-$ with $P_m = 0$ for $m \leq 0$. This implies that the sequence $(A_m)_{m\in\mathbb{Z}}$ does not admit a nonuniform exponential dichotomy on $\mathbb{Z}$.

Example 2.2 (see [9])**.** Given $\omega < 0$ and $\varepsilon \geq 0$ such that $\omega + \varepsilon < 0$, consider the matrices

$$A_m = \begin{pmatrix} e^{\omega+\varepsilon[(-1)^m m-1/2]} & 0 \\ 0 & e^{-\omega+\varepsilon[(-1)^{m+1} m-1/2]} \end{pmatrix}$$

and the projections

$$P_m(x,y) = (x,0) \quad \text{and} \quad Q_m(x,y) = (0,y)$$

for $m \in \mathbb{Z}$. Then the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a nonuniform exponential dichotomy on $\mathbb{Z}$.

The following result gives a necessary and sufficient condition so that a sequence of linear operators admitting nonuniform exponential dichotomies both on $\mathbb{Z}_0^+$ and on $\mathbb{Z}_0^-$ also admits a nonuniform exponential dichotomy on $\mathbb{Z}$.

Theorem 2.3. *A sequence $(A_m)_{m\in\mathbb{Z}} \subset B(X)$ admits a nonuniform exponential dichotomy on $\mathbb{Z}$ if and only if there exist projections P_m^+ for $m \geq 0$ and projections P_m^- for $m \leq 0$ such that:*

1. *$(A_m)_{m\geq 0}$ admits a nonuniform exponential dichotomy on $\mathbb{Z}_0^+$ with projections P_m^+;*
2. *$(A_m)_{m\leq 0}$ admits a nonuniform exponential dichotomy on $\mathbb{Z}_0^-$ with projections P_m^-;*
3. *$X = \operatorname{Im} P_0^+ \oplus \operatorname{Ker} P_0^-$.*

Proof. It is clear that properties 1–3 hold for any sequence $(A_m)_{m\in\mathbb{Z}}$ that admits a nonuniform exponential dichotomy on $\mathbb{Z}$.

Now we prove the converse. We divide the proof into steps.

Step 1. *Construction of Lyapunov norms.* Assume that properties 1–3 hold. Without loss of generality, one can assume that the constants in the notion of a nonuniform exponential dichotomy are the same for both dichotomies (on $\mathbb{Z}_0^+$ and on $\mathbb{Z}_0^-$). Namely, there exist constants $D, \lambda > 0$ and $\varepsilon \geq 0$ such that

$$\begin{aligned}
\|\mathcal{A}(n,m)P_m^+\| &\leq De^{-\lambda(n-m)+\varepsilon m} \quad \text{for} \quad n \geq m \geq 0,\\
\|\mathcal{A}(n,m)Q_m^+\| &\leq De^{-\lambda(m-n)+\varepsilon m} \quad \text{for} \quad 0 \leq n \leq m,\\
\|\mathcal{A}(n,m)P_m^-\| &\leq De^{-\lambda(n-m)+\varepsilon|m|} \quad \text{for} \quad 0 \geq n \geq m,\\
\|\mathcal{A}(n,m)Q_m^-\| &\leq De^{-\lambda(m-n)+\varepsilon|m|} \quad \text{for} \quad n \leq m \leq 0,
\end{aligned} \tag{9}$$

where $Q_m^+ = \mathrm{Id} - P_m^+$ and $Q_m^- = \mathrm{Id} - P_m^-$.

Now we introduce a sequence of Lyapunov norms. For each $n \in \mathbb{Z}$ and $x \in X$, let

$$\|x\|_n = \begin{cases} \|x\|_n^+ & \text{if } n \geq 0,\\ \|x\|_n^- & \text{if } n < 0, \end{cases} \tag{10}$$

where

$$\|x\|_m^+ = \sup_{n \geq m} \left(\|\mathcal{A}(n,m)P_m^+x\|e^{\lambda(n-m)}\right) + \sup_{0 \leq n \leq m} \left(\|\mathcal{A}(n,m)Q_m^+x\|e^{\lambda(m-n)}\right)$$

and

$$\|x\|_m^- = \sup_{0 \geq n \geq m} \left(\|\mathcal{A}(n,m)P_m^-x\|e^{\lambda(n-m)}\right) + \sup_{n \leq m} \left(\|\mathcal{A}(n,m)Q_m^-x\|e^{\lambda(m-n)}\right).$$

For $m \geq 0$ and $x \in X$, by (9) we have

$$\|x\| \leq \|x\|_m^+ \leq 2De^{\varepsilon m}\|x\|. \tag{11}$$

Indeed, by definition,

$$\|x\|_m^+ \geq \|P_m^+x\| + \|Q_m^+x\| \geq \|P_m^+x + Q_m^+x\| = \|x\|$$

and, analogously,

$$\|x\|_m^- \geq \|P_m^-x\| + \|Q_m^-x\| \geq \|P_m^-x + Q_m^-x\| = \|x\|.$$

These inequalities together with (10) yield the first inequality in (11). For the second inequality, using (9) we obtain

$$\|x\|_m^+ \leq De^{\varepsilon m}\|x\| + De^{\varepsilon m}\|x\| = 2De^{\varepsilon m}\|x\|$$

and

$$\|x\|_m^- \leq De^{\varepsilon|m|}\|x\| + De^{\varepsilon|m|}\|x\| = 2De^{\varepsilon|m|}\|x\|.$$

The second inequality in (11) follows now readily from (10).

Moreover,

$$\|\mathcal{A}(n,m)P_m^+x\|_n^+ \leq e^{-\lambda(n-m)}\|x\|_m^+ \quad \text{for} \quad n \geq m \geq 0 \tag{12}$$

and

$$\|\mathcal{A}(n,m)Q_m^+x\|_n^+ \leq e^{-\lambda(m-n)}\|x\|_m^+ \quad \text{for} \quad 0 \leq n \leq m. \tag{13}$$

Similarly, for $m \leq 0$ and $x \in X$, by (9) we have

$$\|x\| \leq \|x\|_m^- \leq 2De^{\varepsilon|m|}\|x\|. \tag{14}$$

Moreover,

$$\|\mathcal{A}(n,m)P_m^-\|_n^- \leq e^{-\lambda(n-m)}\|x\|_m^- \quad \text{for} \quad 0 \geq n \geq m \tag{15}$$

and

$$\|\mathcal{A}(n,m)Q_m^-x\|_n^- \leq e^{-\lambda(m-n)}\|x\|_m^- \quad \text{for} \quad n \leq m \leq 0. \tag{16}$$

It follows from (11) and (14) that

$$\|x\| \le \|x\|_n \le 2De^{\varepsilon|n|}\|x\| \quad \text{for} \quad n \in \mathbb{Z},\ x \in X. \tag{17}$$

Step 2. *Strategy of the proof.* Let

$$Y = \left\{\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \subset X : \sup_{n\in\mathbb{Z}}\|x_n\|_n < +\infty\right\}. \tag{18}$$

The following result was proved in [1].

Lemma 2.4. *Assume that for each $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in Y$, there exists a unique $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y$ satisfying*

$$x_{n+1} - A_n x_n = y_{n+1} \quad \textit{for} \quad n \in \mathbb{Z}. \tag{19}$$

Then the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a nonuniform exponential dichotomy on $\mathbb{Z}$ with respect to the sequence of norms $\|\cdot\|_n$ with $\varepsilon = 0$.

In view of Lemma 2.4 and (17), in order to prove the theorem it is sufficient to show that for each $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in Y$, there exists a unique $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y$ satisfying (19). Indeed, if (19) holds, then by Lemma 2.4 there exist projections P_m for $m \in \mathbb{Z}$ satisfying (2), and inequalities (5) and (6) hold with $\varepsilon = 0$, that is,

$$\|\mathcal{A}(n,m)P_m x\|_n \le De^{-\lambda(n-m)}\|x_m\|_m$$

for $n \ge m$ and

$$\|\mathcal{A}(n,m)Q_m x\|_n \le De^{-\lambda(m-n)}\|x_m\|_m$$

for $n \le m$, where $Q_m = \mathrm{Id} - P_m$. In view of (17) this implies that

$$\|\mathcal{A}(n,m)P_m\| \le D^2 e^{-\lambda(n-m)+\varepsilon|m|} \tag{20}$$

for $n \ge m$ and

$$\|\mathcal{A}(n,m)Q_m\| \le D^2 e^{-\lambda(m-n)+\varepsilon|m|} \tag{21}$$

for $n \le m$. In other words, the sequence $(A_n)_{n\in\mathbb{Z}}$ admits a nonuniform exponential dichotomy on $\mathbb{Z}$.

Step 3. *Ranges of the projections.* The next step is to describe the ranges of the projections P_0^+ and Q_0^-.

Lemma 2.5. *We have*

$$\mathrm{Im}\, P_0^+ = \left\{x \in X : \sup_{m\ge 0}\|\mathcal{A}(m,0)x\|_m^+ < +\infty\right\}.$$

Proof of the lemma. It follows readily from (12) that

$$\sup_{m\ge 0}\|\mathcal{A}(m,0)x\|_m^+ < +\infty \tag{22}$$

for $x \in \mathrm{Im}\, P_0^+$. Now take $x \in X$ satisfying (22). Since $x = P_0^+x + Q_0^+x$, it follows from (12) that

$$\begin{aligned}\sup_{m\ge 0}\|\mathcal{A}(m,0)Q_0^+x\|_m^+ &= \sup_{m\ge 0}\|\mathcal{A}(m,0)(x - P_0^+x)\|_m^+ \\ &\le \sup_{m\ge 0}\|\mathcal{A}(m,0)x\|_m^+ + \sup_{m\ge 0}\|\mathcal{A}(m,0)P_0^+x\|_m^+ < +\infty.\end{aligned}$$

On the other hand, by (13), we have

$$\|Q_0^+x\|_0^+ = \|\mathcal{A}(0,m)\mathcal{A}(m,0)Q_0^+x\|_0^+ \le e^{-\lambda m}\|\mathcal{A}(m,0)Q_0^+x\|_m^+$$

for $m \ge 0$. Letting $m \to \infty$ yields that $Q_0^+x = 0$, i.e., $x = P_0^+x \in \mathrm{Im}\, P_0^+$. $\square$

Lemma 2.6. *For each $m \le 0$, the set $\operatorname{Im} Q_m^-$ consists of all $x \in X$ for which there exists a sequence $(x_n)_{n \le m}$ such that $x_m = x$, $x_{n+1} = A_n x_n$ for $n \le m-1$ and $\sup_{n \le m} \|x_n\|_n^- < +\infty$.*

Proof of the lemma. Clearly, each $x \in \operatorname{Im} Q_m^-$ has the property in the lemma. Conversely, take $x \in X$ with that property and write $x_n = P_n^- x_n + Q_n^- x_n$ for $n \le m$. By (2) we have

$$\mathcal{A}(m,n) P_n^- x_n = P_m^- x_m \quad \text{and} \quad \mathcal{A}(m,n) Q_n^- x_n = Q_m^- x_m$$

for $n \le m$. Hence, it follows from (16) that $\sup_{n \le m} \|P_n^- x_n\|_n^- < +\infty$. On the other hand, by (15), for $n \le m$ we have

$$\|P_m^- x_m\|_m^- = \|\mathcal{A}(m,n) P_n^- x_n\|_n^- \le e^{-\lambda(m-n)} \|P_n^- x_n\|_n^-.$$

Letting $n \to -\infty$ yields that $P_m^- x_m = 0$ and so $x_m = Q_m^- x_m \in \operatorname{Im} Q_m^-$. □

Step 4. *Existence of solutions.* Let

$$Y^+ = \left\{ \mathbf{x} = (x_n)_{n \ge 0} \subset X : \sup_{n \ge 0} \|x_n\|_n^+ < +\infty \right\}.$$

Lemma 2.7. *For each $\mathbf{y} = (y_n)_{n \ge 0} \in Y^+$ with $y_0 = 0$, there exists $\mathbf{x} = (x_n)_{n \ge 0} \in Y^+$ with $x_0 \in \operatorname{Im} Q_0^-$ such that*

$$x_{n+1} - A_n x_n = y_{n+1} \quad \textit{for} \quad n \ge 0. \tag{23}$$

Proof of the lemma. For each $n \ge 0$, let

$$x_n^* = \sum_{k=0}^{n} \mathcal{A}(n,k) P_k^+ y_k - \sum_{k=n+1}^{\infty} \mathcal{A}(n,k) Q_k^+ y_k.$$

It follows from (12) and (13) that

$$\begin{aligned} \|x_n^*\|_n^+ &\le \sum_{k=0}^{n} e^{-\lambda(n-k)} \|y_k\|_k^+ + \sum_{k=n+1}^{\infty} e^{-\lambda(k-n)} \|y_k\|_k^+ \\ &\le \frac{1+e^{-\lambda}}{1-e^{-\lambda}} \sup_{k \ge 0} \|y_k\|_k^+ \end{aligned}$$

for $n \ge 0$ and hence $\mathbf{x}^* = (x_n^*)_{n \ge 0} \in Y^+$. By property 3, one can write $x_0^* = x_0' + x_0''$ with $x_0' \in \operatorname{Im} P_0^+$ and $x_0'' \in \operatorname{Im} Q_0^-$. Let

$$x_n = x_n^* - \mathcal{A}(n,0) x_0' \quad \text{for} \quad n \ge 0.$$

Then $\mathbf{x} = (x_n)_{n \ge 0} \in Y^+$ and $x_0 \in \operatorname{Im} Q_0^-$. Moreover, it is easy to verify that (23) holds. □

Take $\mathbf{y} = (y_n)_{n \in \mathbb{Z}} \in Y$ with $y_n = 0$ for $n \le 0$. By Lemma 2.7, there exists $\mathbf{x}^* = (x_n^*)_{n \ge 0} \in Y^+$ such that $x_0^* \in \operatorname{Im} Q_0^-$ and

$$x_{n+1}^* - A_n x_n^* = y_{n+1} \quad \text{for} \quad n \ge 0.$$

Let

$$x_n = \begin{cases} x_n^* & \text{if } n \ge 0, \\ \mathcal{A}(n,0) x_0^* & \text{if } n < 0. \end{cases} \tag{24}$$

Clearly, $\mathbf{x} = (x_n)_{n \in \mathbb{Z}} \in Y$ and (19) holds.

Now let

$$Y^- = \left\{ \mathbf{x} = (x_n)_{n\le 0} \subset X : \sup_{n\le 0} \|x_n\|_n^- < +\infty \right\}.$$

Lemma 2.8. *For each $\mathbf{y} = (y_n)_{n\le 0} \in Y^-$, there exists $\mathbf{x} = (x_n)_{n\le 0} \in Y^-$ with $x_0 \in \operatorname{Im} P_0^+$ such that*

$$x_{n+1} - A_n x_n = y_{n+1} \quad \text{for} \quad n \le -1. \tag{25}$$

Proof of the lemma. For each $n \le 0$, let

$$x_n^* = -\sum_{k=n+1}^{0} \mathcal{A}(n,k) Q_k^- y_k + \sum_{k=-\infty}^{n} \mathcal{A}(n,k) P_k^- y_k.$$

It follows from (15) and (16) that $\mathbf{x}^* = (x_n^*)_{n\le 0} \in Y^-$. By property 3, one can write $x_0^* = x_0' + x_0''$, with $x_0' \in \operatorname{Im} P_0^+$ and $x_0'' \in \operatorname{Im} Q_0^-$. Let

$$x_n = x_n^* - \mathcal{A}(n,0) x_0'' \quad \text{for} \quad n \le 0.$$

Then $\mathbf{x} = (x_n)_{n\le 0} \in Y^-$ and $x_0 \in \operatorname{Im} P_0^+$. Moreover, it is easy to verify that (25) holds. □

Take $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in Y$ with $y_n = 0$ for $n > 0$. By Lemma 2.8, there exists $\mathbf{x}^* = (x_n^*)_{n\le 0} \in Y^-$ such that $x_0^* \in \operatorname{Im} P_0^+$ and

$$x_{n+1}^* - A_n x_n^* = y_{n+1} \quad \text{for} \quad n \le -1.$$

Let

$$x_n = \begin{cases} x_n^* & \text{if } n \le 0, \\ \mathcal{A}(n,0) x_0^* & \text{if } n > 0. \end{cases} \tag{26}$$

Clearly, $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y$ and (19) holds.

Finally, we note that each sequence $\mathbf{y} \in Y$ can be written in the form $\mathbf{y} = \mathbf{y}^1 + \mathbf{y}^2$ with $\mathbf{y}^1, \mathbf{y}^2 \in Y$ such that $y_n^1 = 0$ for $n \le 0$ and $y_n^2 = 0$ for $n > 0$. Hence, one can obtain a solution of (19) by adding the solutions in (24) and (26).

Step 5. *Uniqueness of solutions.* In order to establish the uniqueness of a solution $\mathbf{x}$ satisfying (19) it is sufficient to consider the case when $\mathbf{y} = 0$. Assume that $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y$ satisfies $x_{n+1} = A_n x_n$ for $n \in \mathbb{Z}$. It follows from Lemmas 2.5 and 2.6 that $x_0 \in \operatorname{Im} P_0^+ \cap \operatorname{Im} Q_0^-$ and thus $x_0 = 0$ (by property 3). Hence, $\mathbf{x} = 0$. This completes the proof of the theorem. □

Now we describe a connection between our work and a result of Pliss in the particular case of uniform exponential dichotomies. We recall that a sequence $(A_m)_{m\in\mathbb{Z}}$ in $B(X)$ admits a *uniform exponential dichotomy on I* if it admits a nonuniform exponential dichotomy on I with $\varepsilon = 0$. The following is a direct consequence of Theorem 2.3 and Lemma 2.4, with the space Y in (18) defined with respect to the norms $\|\cdot\|_n = \|\cdot\|$ for $n \in \mathbb{Z}$.

Theorem 2.9. *The following statements are equivalent:*

1. *for each $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in Y$, there exists a unique $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y$ satisfying (19);*
2. *there exist projections P_m^+ for $m \ge 0$ and P_m^- for $m \le 0$ such that:*
   (a) $(A_m)_{m\ge 0}$ admits a uniform exponential dichotomy on $\mathbb{Z}_0^+$ with projections P_m^+;

(b) $(A_m)_{m\le 0}$ *admits a uniform exponential dichotomy on* $\mathbb{Z}_0^-$ *with projections* P_m^-*;*
(c) $X = \operatorname{Im} P_0^+ \oplus \operatorname{Ker} P_0^-$.

In the finite-dimensional setting, an analogue of Theorem 2.9 was established earlier by Pliss [31] in the case of continuous time. The following is a version of his result for discrete time (see [30, 33] for details).

Theorem 2.10. *Let* X *be a finite-dimensional vector space. The following statements are equivalent:*

1. *for each* $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in Y$*, there exists* $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y$ *satisfying* (19)*;*
2. *there exist projections* P_m^+ *for* $m \ge 0$ *and* P_m^- *for* $m \le 0$ *such that:*
   (a) $(A_m)_{m\ge 0}$ *admits a uniform exponential dichotomy on* $\mathbb{Z}_0^+$ *with projections* P_m^+*;*
   (b) $(A_m)_{m\le 0}$ *admits a uniform exponential dichotomy on* $\mathbb{Z}_0^-$ *with projections* P_m^-*;*
   (c) $X = \operatorname{Im} P_0^+ + \operatorname{Ker} P_0^-$.

Notice that in Theorem 2.10 one does not require the uniqueness of the solution of equation (19) and that the spaces $\operatorname{Im} P_0^+$ and $\operatorname{Ker} P_0^-$ are only required to be transverse. Of course, this causes that neither of the Theorems 2.9 and 2.10 is an automatic consequence of the other.

Related results involving the Fredholm properties of the operator defined by equation (19) (that is, by the uniqueness of its solution) were obtained by Palmer in [25, 26] (see [18, 19] for the case of continuous time).

2.2. **Strong nonuniform exponential dichotomies.** In this section we consider the notion of a strong nonuniform exponential dichotomy.

Let $(A_m)_{m\in I}$ be a sequence of invertible operators in $B(X)$. We define $\mathcal{A}(n,m)$ by (1) for $n \ge m$ and by

$$\mathcal{A}(n,m) = \mathcal{A}(m,n)^{-1} = A_n^{-1}\cdots A_{m-1}^{-1}$$

for $n < m$. We say that $(A_m)_{m\in I}$ admits a *strong nonuniform exponential dichotomy on* I if there exist projections $P_m \in B(X)$ for $m \in I$ satisfying (2) and there exist constants

$$\underline{\lambda} \le \overline{\lambda} < 0 < \underline{\mu} \le \overline{\mu} \quad \text{and} \quad D > 0$$

such that

$$\|\mathcal{A}(m,n)P_n\| \le De^{\overline{\lambda}(m-n)+\varepsilon|n|},$$
$$\|\mathcal{A}(n,m)Q_m\| \le De^{-\underline{\mu}(m-n)+\varepsilon|m|}$$

for $m \ge n$ and

$$\|\mathcal{A}(m,n)P_n\| \le De^{\underline{\lambda}(m-n)+\varepsilon|n|},$$
$$\|\mathcal{A}(n,m)Q_m\| \le De^{-\overline{\mu}(m-n)+\varepsilon|m|}$$

for $m \le n$, where $Q_m = \operatorname{Id} - P_m$.

Example 2.11. Given $\omega < 0$ and $\varepsilon > 0$ such that $\omega + \varepsilon < 0$, consider the real numbers

$$A_m = \begin{cases} e^{\omega+\varepsilon[(-1)^m m - 1/2]} & \text{if } m \ge 0, \\ e^{-\omega+\varepsilon[(-1)^{m+1} m - 1/2]} & \text{if } m < 0. \end{cases}$$

By (7) with $c=0$, for $m \ge n \ge 0$, we have $\mathcal{A}(m,n) \le e^{\omega(m-n)+\varepsilon n}$. Moreover, for $0 \le m \le n$,

$$\begin{aligned}\mathcal{A}(m,n) &= e^{(-\omega+\varepsilon/2)(n-m)-\varepsilon\sum_{k=m}^{n-1}(-1)^k k}\\ &= e^{(-\omega+\varepsilon/2)(n-m)+\varepsilon(-1)^{m-1}\lfloor m/2\rfloor-\varepsilon(-1)^{n-1}\lfloor n/2\rfloor}\\ &\le e^{\omega(m-n)+\varepsilon n}\end{aligned}$$

(see [9]). Now we consider nonpositive times. By (8) with $c=0$, for $m \le n \le 0$, we have $\mathcal{A}(m,n) \le e^{\varepsilon+(\omega+\varepsilon)(n-m)+\varepsilon|n|}$. Moreover, for $n \le m \le 0$,

$$\begin{aligned}\mathcal{A}(m,n) &= e^{-(\omega+\varepsilon/2)(m-n)-\varepsilon\sum_{k=n}^{m-1}(-1)^k k}\\ &\le e^{-(\omega+\varepsilon/2)(m-n)+\varepsilon/2(|n|+|m|+2)}\\ &\le e^{\varepsilon-\omega(m-n)+\varepsilon|n|}\end{aligned}$$

(see [9]). This shows that the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a strong nonuniform exponential dichotomy on $\mathbb{Z}_0^+$ with projections $P_n = \mathrm{Id}$ for $n \ge 0$ and a strong nonuniform exponential dichotomy on $\mathbb{Z}_0^-$ with projections $P_n = 0$ for $n \le 0$. Hence, the sequence does not admit a strong nonuniform exponential dichotomy on $\mathbb{Z}$.

We note that the nonuniform exponential dichotomies in Example 2.1 are not strong, due to the presence of the squares in the formulas for $\mathcal{A}(m,n)$ (see (7) and (8)). On the other hand, it is shown in [9] that the sequence $(A_m)_{m\in\mathbb{Z}}$ in Example 2.2 admits a strong nonuniform exponential dichotomy.

The following result is a version of Theorem 2.3 for strong nonuniform exponential dichotomies. It gives a necessary and sufficient condition so that a sequence admitting strong nonuniform exponential dichotomies both on $\mathbb{Z}_0^+$ and $\mathbb{Z}_0^-$ also admits a strong nonuniform exponential dichotomy on $\mathbb{Z}$.

Theorem 2.12. *A sequence $(A_m)_{m\in\mathbb{Z}} \subset B(X)$ of invertible operators admits a strong nonuniform exponential dichotomy on $\mathbb{Z}$ if and only if there exist projections P_m^+ for $m \ge 0$ and projections P_m^- for $m \le 0$ such that:*

1. *$(A_m)_{m\ge 0}$ admits a strong nonuniform exponential dichotomy on $\mathbb{Z}_0^+$ with projections P_m^+;*
2. *$(A_m)_{m\le 0}$ admits a strong nonuniform exponential dichotomy on $\mathbb{Z}_0^-$ with projections P_m^-;*
3. *$X = \operatorname{Im} P_0^+ \oplus \operatorname{Ker} P_0^-$.*

Proof. It is clear that properties 1–3 hold for any sequence $(A_m)_{m\in\mathbb{Z}}$ that admits a strong nonuniform exponential dichotomy on $\mathbb{Z}$. Now we prove the converse. Assume that properties 1–3 hold. Similarly, without loss of generality, one can assume that the constants in the notion of a strong nonuniform exponential dichotomy are the same for both dichotomies (on $\mathbb{Z}_0^+$ and on $\mathbb{Z}_0^-$).

For each $n \in \mathbb{Z}$ and $x \in X$, we consider the norm

$$\|x\|_n = \begin{cases}\|x\|_n^+ & \text{if } n \ge 0,\\ \|x\|_n^- & \text{if } n < 0,\end{cases}$$

where $\|x\|_m^+$ is the maximum of

$$\sup_{n\ge m}\left(\|\mathcal{A}(n,m)P_m^+x\|e^{-\overline{\lambda}(n-m)}\right) + \sup_{0\le n\le m}\left(\|\mathcal{A}(n,m)P_m^+x\|e^{-\underline{\lambda}(n-m)}\right)$$

and

$$\sup_{0\le n\le m}\left(\|\mathcal{A}(n,m)Q_m^+x\|e^{\underline{\mu}(m-n)}\right)+\sup_{n\ge m}\left(\|\mathcal{A}(n,m)Q_m^+x\|e^{\overline{\mu}(m-n)}\right),$$

and where $\|x\|_m^-$ is the maximum of

$$\sup_{0\ge n\ge m}\left(\|\mathcal{A}(n,m)P_m^-x\|e^{-\overline{\lambda}(n-m)}\right)+\sup_{n\le m}\left(\|\mathcal{A}(n,m)P_m^-x\|e^{-\underline{\lambda}(n-m)}\right)$$

and

$$\sup_{n\le m}\left(\|\mathcal{A}(n,m)Q_m^-x\|e^{\underline{\mu}(m-n)}\right)+\sup_{0\ge n\ge m}\left(\|\mathcal{A}(n,m)Q_m^-x\|e^{\overline{\mu}(m-n)}\right).$$

One can easily verify that

$$\|x\|\le\|x\|_m^+\le 2De^{\varepsilon m}\|x\| \quad \text{for} \quad x\in X,\ m\ge 0. \tag{27}$$

Moreover, following arguments in [5] yields that

$$\begin{aligned}\|\mathcal{A}(m,n)P_n^+x\|_m^+&\le 2e^{\overline{\lambda}(m-n)}\|x\|_n^+\\ \|\mathcal{A}(n,m)Q_m^+x\|_n&\le 2e^{-\underline{\mu}(m-n)}\|x\|_m^+\end{aligned} \tag{28}$$

for $m\ge n\ge 0$ and

$$\begin{aligned}\|\mathcal{A}(m,n)P_n^+x\|_m^+&\le 2e^{\underline{\lambda}(m-n)}\|x\|_n^+,\\ \|\mathcal{A}(n,m)Q_m^+x\|_n^+&\le 2e^{-\overline{\mu}(m-n)}\|x\|_m^+\end{aligned} \tag{29}$$

for $0\le m\le n$. By (28) and (29) we have

$$\|A_nx\|_{n+1}^+\le\|A_nP_n^+x\|_{n+1}^++\|A_nQ_n^+x\|_{n+1}^+\le 4e^{\overline{\mu}}\|x\|_n^+$$

and similarly,

$$\|A_n^{-1}x\|_n^+\le 4e^{-\underline{\lambda}}\|x\|_{n+1}^+$$

for $x\in X$ and $n\ge 0$. Hence,

$$\frac{1}{4}e^{\underline{\lambda}}\|x\|_n^+\le\|A_nx\|_{n+1}^+\le 4e^{\overline{\mu}}\|x\|_n^+ \quad \text{for} \quad x\in X,\ n\ge 0. \tag{30}$$

Analogously, one can easily verify that

$$\|x\|\le\|x\|_m^-\le 2De^{\varepsilon|m|}\|x\| \quad \text{for} \quad x\in X,\ m\le 0. \tag{31}$$

Moreover,

$$\begin{aligned}\|\mathcal{A}(m,n)P_n^-x\|_m^-&\le 2e^{\overline{\lambda}(m-n)}\|x\|_n^-,\\ \|\mathcal{A}(n,m)Q_m^-x\|_n^-&\le 2e^{-\underline{\mu}(m-n)}\|x\|_m^-\end{aligned} \tag{32}$$

for $0\ge m\ge n$ and

$$\begin{aligned}\|\mathcal{A}(m,n)P_n^-x\|_m^-&\le 2e^{\underline{\lambda}(m-n)}\|x\|_n^-,\\ \|\mathcal{A}(n,m)Q_m^-x\|_n^-&\le 2e^{-\overline{\mu}(m-n)}\|x\|_m^-\end{aligned}$$

for $m\le n\le 0$. This implies that

$$\frac{1}{4}e^{\underline{\lambda}}\|x\|_n^-\le\|A_nx\|_{n+1}^-\le 4e^{\overline{\mu}}\|x\|_n^- \quad \text{for} \quad x\in X,\ n\le -1. \tag{33}$$

By (27) and (31) we have

$$\|x\|\le\|x\|_m\le 2De^{\varepsilon|m|}\|x\| \quad \text{for} \quad x\in X,\ m\in\mathbb{Z}. \tag{34}$$

Moreover, by (30) and (33) (together with the fact that the norms $\|\cdot\|_0^+$ and $\|\cdot\|_0^-$ are equivalent) there exist constants $C_1, C_2 > 1$ such that

$$\frac{1}{C_2}\|x\|_n \le \|A_n x\|_{n+1} \le C_1\|x\|_n \quad \text{for} \quad x \in X,\ n \in \mathbb{Z}. \tag{35}$$

Indeed, it follows from (30) and (33) that (35) holds for $n \ge 0$ and $n < -1$ with $C_1 = 4e^{\overline{\mu}}$ and $C_2 = 4e^{-\underline{\lambda}}$. On the other hand, for $n = -1$, using also (27) and (31) we obtain

$$\begin{aligned}\|A_{-1}x\|_0 &= \|A_{-1}x\|_0^+ \le D\|A_{-1}x\| \\ &\le D\|A_{-1}x\|_0^- \le 4De^{\overline{\mu}}\|x\|_{-1}\end{aligned}$$

and, similarly,

$$\begin{aligned}\|A_{-1}x\|_0 &= \|A_{-1}x\|_0^+ \ge \|A_{-1}x\| \\ &\ge \frac{1}{D}\|A_{-1}x\|_0^- \ge \frac{1}{4D}e^{\underline{\lambda}}\|x\|_{-1}.\end{aligned}$$

Hence, (35) holds for $n \in \mathbb{Z}$ with $C_1 = 4De^{\overline{\mu}}$ and $C_2 = 4De^{-\underline{\lambda}}$.

In a similar manner to that in the proof of Theorem 2.3, inequalities (28) and (32) imply that the sequence $(A_n)_{n\in\mathbb{Z}}$ admits a nonuniform exponential dichotomy with respect to the sequence of norms $\|\cdot\|_n$ with $\varepsilon = 0$. Hence, there exist projections P_n for $n \in \mathbb{Z}$ satisfying (2) and there exist constants $C, \lambda > 0$ such that

$$\begin{aligned}\|\mathcal{A}(m,n)P_n x\|_m &\le Ce^{-\lambda(m-n)}\|x\|_n, \\ \|\mathcal{A}(n,m)Q_m x\|_n &\le Ce^{-\lambda(m-n)}\|x\|_m\end{aligned} \tag{36}$$

for $m \ge n$. Finally, by (34), (35) and (36) we conclude that

$$\begin{aligned}\|\mathcal{A}(m,n)P_n x\| &\le CDe^{-\lambda(m-n)+\varepsilon|n|}\|x\|, \\ \|\mathcal{A}(n,m)Q_m x\| &\le CDe^{-\lambda(m-n)+\varepsilon|m|}\|x\|\end{aligned}$$

for $m \ge n$ and

$$\begin{aligned}\|\mathcal{A}(m,n)P_n x\| &\le CDe^{(\log C_2)(n-m)+\varepsilon|n|}\|x\|, \\ \|\mathcal{A}(n,m)Q_m x\| &\le CDe^{(\log C_1)(n-m)+\varepsilon|m|}\|x\|\end{aligned}$$

for $m \le n$. Therefore, the sequence $(A_n)_{n\in\mathbb{Z}}$ admits a strong nonuniform exponential dichotomy on $\mathbb{Z}$. This completes the proof of the theorem. ☐

3. **Continuous time.**

In this section we obtain corresponding results to those in Section 2 for continuous time.

3.1. **Nonuniform exponential dichotomies.**

We continue to denote by $B(X)$ the set of all bounded linear operators on a Banach space X. Let $I \in \{\mathbb{R}, \mathbb{R}_0^+, \mathbb{R}_0^-\}$ be an interval. A family $T(t,\tau)$ for $t, \tau \in I$ with $t \ge \tau$ of linear operators in $B(X)$ is said to be an *evolution family on I* if:

1. $T(t,t) = \mathrm{Id}$ for $t \in I$;
2. $T(t,s)T(s,\tau) = T(t,\tau)$ for $t, s, \tau \in I$ with $t \ge s \ge \tau$;
3. for each $t, \tau \in I$ and $x \in X$, the map $s \mapsto T(t,s)x$ is continuous on $(-\infty, t] \cap I$ and the map $s \mapsto T(s,\tau)x$ is continuous on $[\tau, \infty) \cap I$.

We say that an evolution family $T(t,\tau)$ on I admits an *nonuniform exponential dichotomy on I* if:

1. there exist projections $P_t \in B(X)$ for $t \in I$ satisfying
$$P_t T(t,\tau) = T(t,\tau)P_\tau \quad \text{for} \quad t \geq \tau \tag{37}$$
such that each map
$$T(t,\tau)|\operatorname{Ker} P_\tau \colon \operatorname{Ker} P_\tau \to \operatorname{Ker} P_t$$
is invertible;
2. there exist constants $\lambda, D > 0$ and $\varepsilon \geq 0$ such that each $t, \tau \in I$ we have
$$\|T(t,\tau)P_\tau\| \leq De^{-\lambda(t-\tau)+\varepsilon|\tau|} \quad \text{for} \quad t \geq \tau \tag{38}$$
and
$$\|T(t,\tau)Q_\tau\| \leq De^{-\lambda(\tau-t)+\varepsilon|\tau|} \quad \text{for} \quad t \leq \tau, \tag{39}$$
where $Q_\tau = \operatorname{Id} - P_\tau$ and
$$T(t,\tau) = (T(\tau,t)|\operatorname{Ker} P_t)^{-1} \colon \operatorname{Ker} P_\tau \to \operatorname{Ker} P_t$$
for $t \leq \tau$.

More generally, given a family of norms $\|\cdot\|_t$ for $t \in I$ on X, we say that $T(t,\tau)$ admits a *nonuniform exponential dichotomy on I with respect to the family of norms* $\|\cdot\|_t$ if conditions 1–2 hold with (38) and (39) replaced respectively by
$$\|T(t,\tau)P_\tau x\|_t \leq De^{-\lambda(t-\tau)+\varepsilon|\tau|}\|x\|_\tau \quad \text{for} \quad t \geq \tau,\ x \in X$$
and
$$\|T(t,\tau)Q_\tau x\|_t \leq De^{-\lambda(\tau-t)+\varepsilon|\tau|}\|x\|_\tau \quad \text{for} \quad t \leq \tau,\ x \in X.$$

Example 3.1. Given $\omega > \varepsilon > 0$ and $c > 0$, consider the evolution family $T_1(t,s)$ on $\mathbb{R}_0^+$ defined by
$$T_1(t,s) = e^{(-\omega+\varepsilon)(t-s)+\varepsilon t(\cos t-1)-\varepsilon s(\cos s-1)+\varepsilon(\sin s-\sin t)-c(t^2-s^2)}.$$
For $t \geq s \geq 0$, we have
$$T_1(t,s) \leq e^{2\varepsilon}e^{(-\omega+\varepsilon)(t-s)+2\varepsilon s}.$$
and thus $T_1(t,s)$ admits a nonuniform exponential dichotomy on $\mathbb{R}_0^+$ with projections $P_t = \operatorname{Id}$ for $t \geq 0$. Now consider the evolution family $T_2(t,s)$ on $\mathbb{R}_0^-$ defined by
$$T_2(t,s) = e^{(\omega-\varepsilon)(t-s)-\varepsilon t(\cos t-1)+\varepsilon s(\cos s-1)-\varepsilon(\sin s-\sin t)-c(t^2-s^2)}.$$
For $t \leq s \leq 0$, we have
$$T_2(t,s) \leq e^{2\varepsilon}e^{(\varepsilon-\omega)(s-t)+2\varepsilon|s|}$$
and thus $T_2(t,s)$ admits a nonuniform exponential dichotomy on $\mathbb{R}_0^-$ with projections $P_t = 0$ for $t \leq 0$. Then the evolution family $T(t,s)$ on $\mathbb{R}$ defined by
$$T(t,s) = \begin{cases} T_1(t,s) & \text{if } t \geq s \geq 0, \\ T_2(t,s) & \text{if } 0 \geq t \geq s, \\ T_1(t,0)T_2(0,s) & \text{if } t > 0 > s \end{cases}$$
does not admit a nonuniform exponential dichotomy on $\mathbb{R}$.

In a similar manner to that in Theorem 2.3, the following result gives a necessary and sufficient condition so that a sequence admitting nonuniform exponential dichotomies both on $\mathbb{R}_0^+$ and $\mathbb{R}_0^-$ also admits a nonuniform exponential dichotomy on $\mathbb{R}$.

Theorem 3.2. *An evolution family $T(t,\tau)$ admits a nonuniform exponential dichotomy on $\mathbb{R}$ if and only if there exist projections P_t^+ for $t \ge 0$ and projections P_t^- for $t \le 0$ such that:*

1. *$T(t,\tau)$ admits a nonuniform exponential dichotomy on $\mathbb{R}_0^+$ with projections P_t^+;*
2. *$T(t,\tau)$ admits a nonuniform exponential dichotomy on $\mathbb{R}_0^-$ with projections P_t^-;*
3. *$X = \operatorname{Im} P_0^+ \oplus \operatorname{Ker} P_0^-$.*

Proof. As in the proof of Theorem 2.3, it is sufficient to show that if properties 1–3 hold, then the evolution family admits a nonuniform exponential dichotomy on $\mathbb{R}$.

We first introduce Lyapunov norms. For $t \ge 0$ and $x \in X$, let

$$\|x\|_t^+ = \sup_{\tau \ge t} \big(\|T(\tau,t)P_t^+ x\| e^{\lambda(\tau-t)}\big) + \sup_{0 \le \tau \le t} \big(\|T(\tau,t)Q_t^+ x\| e^{\lambda(t-\tau)}\big).$$

It follows from (38) and (39) that

$$\|x\| \le \|x\|_t^+ \le De^{\varepsilon t}\|x\| \quad \text{for} \quad t \ge 0,\ x \in X. \tag{40}$$

Moreover,

$$\|T(t,\tau)P_\tau^+ x\|_t^+ \le e^{-\lambda(t-\tau)}\|x\|_\tau^+ \tag{41}$$

for $t \ge \tau \ge 0$ and $x \in X$, and similarly,

$$\|T(t,\tau)P_\tau^+ x\|_t^+ \le e^{-\lambda(\tau-t)}\|x\|_\tau^+ \tag{42}$$

for $0 \le t \le \tau$ and $x \in X$.

On the other hand, for $t \le 0$ and $x \in X$, let

$$\|x\|_t^- = \sup_{0 \ge \tau \ge t} \big(\|T(\tau,t)P_t^- x\| e^{\lambda(\tau-t)}\big) + \sup_{\tau \le t} \big(\|T(\tau,t)Q_t^- x\| e^{\lambda(t-\tau)}\big).$$

It follows from (38) and (39) that

$$\|x\| \le \|x\|_t^- \le De^{\varepsilon|t|}\|x\| \quad \text{for} \quad t \le 0,\ x \in X. \tag{43}$$

Moreover,

$$\|T(t,\tau)P_\tau^- x\|_t^- \le e^{-\lambda(t-\tau)}\|x\|_t^- \tag{44}$$

for $0 \ge t \ge \tau$ and $x \in X$, and similarly,

$$\|T(t,\tau)Q_\tau^- x\|_t^- \le e^{-\lambda(\tau-t)}\|x\|_\tau^- \tag{45}$$

for $t \le \tau \le 0$ and $x \in X$. In addition, one can show that

$$s \mapsto \|T(s,t)x\|_s^+ \text{ is continuous on } [t,+\infty)$$

for $t \ge 0$ and $x \in X$, and that

$$s \mapsto \|T(s,t)x\|_s^- \text{ is continuous on } [t,0]$$

for $t \le 0$ and $x \in X$ (see [2] for a detailed argument). Finally, for $t \in \mathbb{R}$ and $x \in X$, let

$$\|x\|_t = \begin{cases} \|x\|_t^+ & \text{if } t \ge 0, \\ \|x\|_t^- & \text{if } t < 0. \end{cases} \tag{46}$$

It follows from (40) and (43) that

$$\|x\| \le \|x\|_t \le De^{\varepsilon|t|}\|x\| \quad \text{for} \quad x \in X,\ t \in \mathbb{R}.$$

Now we consider the spaces

$$Y = \left\{ x\colon \mathbb{R} \to X \text{ continuous} : \sup_{t \in \mathbb{R}} \|x(t)\|_t < +\infty \right\}$$

and

$$Y_1 = \left\{ x\colon \mathbb{R} \to X \text{ measurable} : \sup_{t\in\mathbb{R}} \int_t^{t+1} \|x(\tau)\|_\tau \, d\tau < +\infty \right\}.$$

The following result is proved in Appendix A (the argument requires additional material that would complicate the exposition at this point).

Lemma 3.3. *Assume that for each $y \in Y_1$, there exists a unique $x \in Y$ satisfying*

$$x(t) = T(t,\tau)x(\tau) + \int_\tau^t T(t,s)y(s)\,ds \quad \textit{for} \quad t \geq \tau. \tag{47}$$

Then $T(t,\tau)$ admits a nonuniform exponential dichotomy with respect to the family of norms $\|\cdot\|_t$ with $\varepsilon = 0$.

In view of Lemma 3.3, in order to prove Theorem 3.2 it is sufficient to show that for each $y \in Y_1$, there exists a unique $x \in Y$ satisfying (47).

The proofs of the following two lemmas are analogous to those of Lemmas 2.5 and 2.6.

Lemma 3.4. *We have*

$$\operatorname{Im} P_0^+ = \left\{ x \in X : \sup_{t\geq 0} \|T(t,0)x\|_t^+ < +\infty \right\}.$$

Lemma 3.5. *For each $t \leq 0$, the set $\operatorname{Im} Q_t^-$ consists of all $x \in X$ for which there exists a continuous function $x\colon (-\infty, t] \to X$ such that $x(t) = x$, $x(s_1) = T(s_1, s_2)x(s_2)$ for $t \geq s_1 \geq s_2$ and $\sup_{s\leq t}\|x(s)\|_s^- < +\infty$.*

Now we introduce auxiliary spaces. Let

$$Y^+ = \left\{ x\colon \mathbb{R}_0^+ \to X \text{ continuous} : \sup_{t\geq 0} \|x(t)\|_t^+ < +\infty \right\}$$

and

$$Y_1^+ = \left\{ x\colon \mathbb{R}_0^+ \to X \text{ measurable} : \sup_{t\geq 0} \int_t^{t+1} \|x(\tau)\|_\tau^+ \, d\tau < +\infty \right\}.$$

Lemma 3.6. *For each $y \in Y_1^+$, there exists $x \in Y^+$ with $x(0) \in \operatorname{Im} Q_0^-$ such that*

$$x(t) = T(t,\tau)x(\tau) + \int_\tau^t T(t,s)y(s)\,ds \quad \textit{for} \quad t \geq \tau \geq 0. \tag{48}$$

Proof of the lemma. Take $y \in Y_1^+$ and extend it to a function $y\colon \mathbb{R} \to X$ by letting $y(t) = 0$ for $t < 0$. Moreover, for $t \geq 0$, let

$$x_1^*(t) = \int_0^t T(t,\tau)P_\tau y(\tau)\,d\tau \quad \text{and} \quad x_2^*(t) = \int_t^\infty T(t,\tau)Q_\tau y(\tau)\,d\tau.$$

It follows from (41) that

$$\begin{aligned}
\|x_1^*(t)\|_t &\leq \int_{-\infty}^t \|T(t,\tau)P_\tau^+ y(\tau)\|_t \, d\tau \\
&\leq \int_{-\infty}^t e^{-\lambda(t-\tau)} \|y(\tau)\|_\tau \, d\tau \\
&= \sum_{m=0}^\infty \int_{t-m-1}^{t-m} e^{-\lambda(t-\tau)} \|y(\tau)\|_\tau \, d\tau
\end{aligned}$$

$$\begin{aligned}&\leq \sum_{m=0}^{\infty} e^{-\lambda m} \int_{t-m-1}^{t-m} \|y(\tau)\|_\tau \, d\tau \\ &\leq \frac{1}{1-e^{-\lambda}} \sup_{t\geq 0} \int_t^{t+1} \|y(\tau)\|_\tau \, d\tau\end{aligned}$$

for $t \geq 0$. Similarly, by (42),

$$\|x_2^*(t)\|_t \leq \frac{1}{1-e^{-\lambda}} \sup_{t\geq 0} \int_t^{t+1} \|y(\tau)\|_\tau \, d\tau$$

for $t \geq 0$. Now let $x^*(t) = x_1^*(t) - x_2^*(t)$. Clearly, $\sup_{t\geq 0}\|x^*(t)\|_t < +\infty$. For $\tau \geq 0$, we have

$$\begin{aligned}x^*(t) &= \int_\tau^t T(t,s)y(s)\, ds - \int_\tau^t T(t,s)P_s^+ y(s)\, ds - \int_\tau^t T(t,s)Q_s^+ y(s)\, ds \\ &\quad + \int_0^t T(t,s)P_s^+ y(s)\, ds - \int_t^\infty T(t,s)Q_s^+ y(s)\, ds \\ &= \int_\tau^t T(t,s)y(s)\, ds + \int_0^\tau T(t,s)P_s^+ y(s)\, ds - \int_\tau^\infty T(t,s)Q_s^+ y(s)\, ds \\ &= \int_\tau^t T(t,s)y(s)\, ds + T(t,\tau)x^*(\tau)\end{aligned}$$

for $t \geq \tau$ and so identity (48) holds with x replaced by x^*. In particular, this implies that x^* is continuous and so $x^* \in Y^+$. By property 3, one can write $x^*(0) = x_0' + x_0''$ with $x_0' \in \operatorname{Im} P_0^+$ and $x_0'' \in \operatorname{Im} Q_0^-$. We define $x\colon \mathbb{R}_0^+ \to X$ by

$$x(t) = x^*(t) - T(t,0)x_0'$$

for $t \geq 0$. Then $x \in Y^+$, $x(0) \in \operatorname{Im} Q_0^-$ and (48) holds. □

Take $y \in Y_1$ with $y(t) = 0$ for $t < 0$. By Lemma 3.6, there exists $x^* \in Y^+$ such that (48) holds and $x^*(0) \in \operatorname{Im} Q_0^-$. Let

$$x(t) = \begin{cases} x^*(t) & \text{if } t \geq 0, \\ T(t,0)x^*(0) & \text{if } t < 0. \end{cases} \tag{49}$$

Clearly, $x \in Y$ and (47) holds.

Similarly, let

$$Y^- = \left\{ x\colon \mathbb{R}_0^- \to X \text{ continuous} : \sup_{t\leq 0} \|x(t)\|_t^- < +\infty \right\}$$

and

$$Y_1^- = \left\{ x\colon \mathbb{R}_0^- \to X \text{ measurable} : \sup_{t\leq 0} \int_{t-1}^t \|x(\tau)\|_\tau^- \, d\tau < +\infty \right\}.$$

Lemma 3.7. *For each $y \in Y_1^-$, there exists $x \in Y^-$ with $x(0) \in \operatorname{Im} P_0^+$ such that*

$$x(t) = T(t,\tau)x(\tau) + \int_\tau^t T(t,s)y(s)\, ds \quad \text{for} \quad 0 \geq t \geq \tau. \tag{50}$$

Proof of the lemma. Take $y \in Y_1^-$. For $t \leq 0$, let

$$x^*(t) = -\int_t^0 T(t,\tau)Q_\tau^- y(\tau)\, d\tau + \int_{-\infty}^t T(t,\tau)P_\tau^- y(\tau)\, d\tau.$$

It follows easily from (44) and (45) that $\sup_{t\le 0}\|x^*(t)\|_t^- < +\infty$. Moreover, it is easy to verify that identity (50) holds with x replaced by x^*. By property 3, one can write $x^*(0) = x_0' + x_0''$ with $x_0' \in \operatorname{Im} P_0^+$ and $x_0'' \in \operatorname{Im} Q_0^-$. We define $x\colon \mathbb{R}_0^- \to X$ by

$$x(t) = x^*(t) - T(t,0)x_0''$$

for $t \le 0$. Then $x \in Y^-$, $x(0) \in \operatorname{Im} P_0^+$ and (50) holds. □

Take $y \in Y_1$ with $y(t) = 0$ for $t \ge 0$. By Lemma 3.7, there exists $x^* \in Y^-$ such that (50) holds and $x^*(0) \in \operatorname{Im} P_0^+$. Let

$$x(t) = \begin{cases} x^*(t) & \text{if } t \le 0, \\ T(t,0)x^*(0) & \text{if } t > 0. \end{cases} \tag{51}$$

Clearly, $x \in Y$ and (47) holds.

Finally, each $y \in Y_1$ can be written in the form $y = y^1 + y^2$ with $y^1, y^2 \in Y_1$ such that $y^1(t) = 0$ for $t \le 0$ and $y^2(t) = 0$ for $t > 0$. Hence, we obtain a solution of (47) by adding the solutions in (49) and (51).

In order to prove the uniqueness of the solution, it is sufficient to consider the case when $y = 0$. Assume that $x \in Y$ satisfies $x(t) = T(t,\tau)x(\tau)$ for $t \ge \tau$. It follows from Lemmas 3.4 and 3.5 that $x(0) \in \operatorname{Im} P_0^+ \cap \operatorname{Im} Q_0^-$ and thus $x(0) = 0$ (by property 3). Hence, $x = 0$. This completes the proof of the theorem. □

3.2. **Strong nonuniform exponential dichotomies.** We say that an invertible evolution family $T(t,\tau)$ for $t,\tau \in I$ admits a *strong nonuniform exponential dichotomy* on I if there exist projections $P_t \in B(X)$ for $t \in I$ satisfying (37) and there exist constants

$$\underline{\lambda} \le \overline{\lambda} < 0 < \underline{\mu} \le \overline{\mu}, \quad \varepsilon \ge 0 \quad \text{and} \quad D > 0$$

such that

$$\|T(t,\tau)P_\tau\| \le De^{\overline{\lambda}(t-\tau)+\varepsilon|\tau|},$$
$$\|T(\tau,t)Q_t\| \le De^{-\underline{\mu}(t-\tau)+\varepsilon|t|}$$

for $t \ge \tau$ and

$$\|T(t,\tau)P_\tau\| \le De^{\underline{\lambda}(t-\tau)+\varepsilon|\tau|},$$
$$\|T(\tau,t)Q_t\| \le De^{-\overline{\mu}(t-\tau)+\varepsilon|t|}$$

for $t \le \tau$, where $Q_\tau = \mathrm{Id} - P_\tau$.

Example 3.8. Given $\omega > \varepsilon > 0$, consider the evolution family $T_1(t,s)$ on $\mathbb{R}_0^+$ defined by

$$T_1(t,s) = e^{(-\omega+\varepsilon)(t-s)+\varepsilon t(\cos t-1)-\varepsilon s(\cos s-1)+\varepsilon(\sin s-\sin t)}.$$

We have

$$T_1(t,s) \le e^{2\varepsilon}e^{(-\omega+\varepsilon)(t-s)+2\varepsilon s}$$

for $t \ge s \ge 0$ and

$$T_1(t,s) \le e^{2\varepsilon}e^{(-\omega+\varepsilon)(t-s)+2\varepsilon s}$$

for $0 \le t \le s$. Now we consider negative times. Namely, we consider the evolution family $T_2(t,s)$ on $\mathbb{R}_0^-$ defined by

$$T_2(t,s) = e^{(\omega-\varepsilon)(t-s)-\varepsilon t(\cos t-1)+\varepsilon s(\cos s-1)-\varepsilon(\sin s-\sin t)}.$$

We have
$$T_2(t,s) \le e^{2\varepsilon} e^{(\varepsilon-\omega)(s-t)+2\varepsilon|s|}$$
for $t \le s \le 0$ and
$$T_2(t,s) \le e^{2\varepsilon} e^{(\omega-\varepsilon)(t-s)+2\varepsilon|s|}$$
for $0 \ge t \ge s$. Then the evolution family $T(t,s)$ on $\mathbb{R}$ defined by
$$T(t,s) = \begin{cases} T_1(t,s) & \text{if } t \ge s \ge 0, \\ T_2(t,s) & \text{if } 0 \ge t \ge s, \\ T_1(t,0)T_2(0,s) & \text{if } t > 0 > s \end{cases}$$
admits both strong nonuniform exponential dichotomies on $\mathbb{R}_0^+$ and on $\mathbb{R}_0^-$, but it does not admit a strong nonuniform exponential dichotomy on $\mathbb{R}$.

The following result is a version of Theorem 2.12 for continuous time.

Theorem 3.9. *An evolution family $T(t,\tau)$ admits a strong nonuniform exponential dichotomy on $\mathbb{R}$ if and only if there exist projections P_t^+ for $t \ge 0$ and projections P_t^- for $t \le 0$ such that:*
1. *$T(t,\tau)$ admits a strong nonuniform exponential dichotomy on $\mathbb{R}_0^+$ with projections P_t^+;*
2. *$T(t,\tau)$ admits a strong nonuniform exponential dichotomy on $\mathbb{R}_0^-$ with projections P_t^-;*
3. *$X = \operatorname{Im} P_0^+ \oplus \operatorname{Ker} P_0^-$.*

Proof. In a similar manner to that in the proof of Theorem 2.12, one can introduce norms $\|\cdot\|_t^+$ for $t \ge 0$ such that
$$\begin{aligned} \|T(t,\tau)P_\tau^+ x\|_t^+ &\le 2e^{\overline{\lambda}(t-\tau)}\|x\|_\tau^+, \\ \|T(\tau,t)Q_t^+ x\|_\tau^+ &\le 2e^{-\underline{\mu}(t-\tau)}\|x\|_t^+ \end{aligned} \tag{52}$$
for $t \ge \tau \ge 0$ and
$$\begin{aligned} \|T(t,\tau)P_\tau^+ x\|_t &\le 2e^{\underline{\lambda}(t-\tau)}\|x\|_\tau^+, \\ \|T(\tau,t)Q_t^+ x\|_\tau^+ &\le 2e^{-\overline{\mu}(t-\tau)}\|x\|_t^+ \end{aligned} \tag{53}$$
for $0 \le t \le \tau$. Then
$$\|x\| \le \|x\|_t^+ \le 2De^{\varepsilon t}\|x\| \quad \text{for} \quad x \in X,\ t \ge 0$$
and it follows from (52) and (53) that there exist $K, a > 0$ such that
$$\|T(t,\tau)x\|_t^+ \le Ke^{a|t-\tau|}\|x\|_\tau^+ \quad \text{for} \quad x \in X,\ t,\tau \ge 0. \tag{54}$$

Similarly, one can introduce norms $\|\cdot\|_t^-$ for $t \le 0$ such that
$$\begin{aligned} \|T(t,\tau)P_\tau^- x\|_t^- &\le 2e^{\overline{\lambda}(t-\tau)}\|x\|_\tau^-, \\ \|T(\tau,t)Q_t^- x\|_\tau^- &\le 2e^{-\underline{\mu}(t-\tau)}\|x\|_t^- \end{aligned} \tag{55}$$
for $0 \ge t \ge \tau$ and
$$\begin{aligned} \|T(t,\tau)P_\tau^- x\|_t^- &\le 2e^{\underline{\lambda}(t-\tau)}\|x\|_\tau^-, \\ \|T(\tau,t)Q_t^- x\|_\tau^- &\le 2e^{-\overline{\mu}(t-\tau)}\|x\|_t^- \end{aligned} \tag{56}$$
for $t \le \tau \le 0$. Then
$$\|x\| \le \|x\|_t^- \le 2De^{\varepsilon|t|}\|x\| \quad \text{for} \quad x \in X,\ t \ge 0$$

and it follows from (55) and (56) that there exist $K', a' > 0$ such that

$$\|T(t,\tau)x\|_t^- \le K' e^{a'|t-\tau|}\|x\|_\tau^- \quad \text{for} \quad x \in X,\ t,\tau \le 0. \tag{57}$$

For each $t \ge 0$ and $x \in X$, let

$$\|x\|_t = \begin{cases} \|x\|_t^+ & \text{if } t \ge 0, \\ \|x\|_t^- & \text{if } t < 0. \end{cases}$$

We have

$$\|x\| \le \|x\|_t \le 2De^{\varepsilon|t|}\|x\| \quad \text{for} \quad x \in X,\ t \in \mathbb{R}. \tag{58}$$

It follows easily from (54) and (57) that there exist $L, b > 0$ such that

$$\|T(t,\tau)x\|_t \le Le^{b|t-\tau|}\|x\|_\tau \quad \text{for} \quad x \in X,\ t,\tau \in \mathbb{R}. \tag{59}$$

Indeed, it follows from (54) and (57) that (59) holds with $L = K$ when $t, \tau \ge 0$ and with $L = K'$ when $t, \tau \le 0$. For $t \ge 0 > \tau$, we have

$$\begin{aligned} \|T(t,\tau)x\|_t = \|T(t,\tau)x\|_t^+ &\le Ke^{at}\|T(0,\tau)x\|_0^+ \\ &\le KDe^{at}\|T(0,\tau)x\|_0^- \le KK'De^{at-a'\tau}\|x\|_\tau \end{aligned}$$

and, similarly,

$$\|T(\tau,t)x\|_\tau \le KK'De^{at-a'\tau}\|x\|_t.$$

Hence, (59) holds with $b = \max\{a, a'\}$ and $L = KK'D$. Moreover, the estimates in (52) and (55) can be used to repeat arguments in the proof of Theorem 3.2 in order to show that the evolution family $T(t,\tau)$ admits a nonuniform exponential dichotomy with respect to a family of norms $\|\cdot\|_t$ with $\varepsilon = 0$. Finally, it follows from (58) and (59) that $T(t,\tau)$ admits a strong nonuniform exponential dichotomy on $\mathbb{R}$. □

4. Lyapunov regular dynamics.

In this section, as an application of the results in the former sections, we give a characterization of the set of Lyapunov exponents of a Lyapunov regular dynamics. We consider both discrete and continuous time.

4.1. Discrete time.

We consider a linear difference equation

$$x_{n+1} = A_n x_n \tag{60}$$

on $\mathbb{R}^d$, where $(A_n)_{n\in\mathbb{Z}}$ is a sequence of invertible $d \times d$ matrices. The dynamics in (60) is said to be *Lyapunov regular* if there exist a decomposition

$$\mathbb{R}^d = \bigoplus_{i=1}^{s} E_i \tag{61}$$

and real numbers $\lambda_1 < \cdots < \lambda_s$ such that:

1. if $i = 1, \ldots, s$ and $v \in E_i \setminus \{0\}$, then

$$\lim_{n\to\pm\infty} \frac{1}{n}\log\|\mathcal{A}(n,0)v\| = \lambda_i; \tag{62}$$

2.

$$\lim_{n\to\pm\infty} \frac{1}{n}\log|\det \mathcal{A}(n,0)| = \sum_{i=1}^{s} \lambda_i \dim E_i.$$

We shall say that a sequence $(A_n)_{n\in I}$ admits a *nonuniform exponential dichotomy on I with an arbitrarily small nonuniform part* if there exist projections P_m for $m \in I$, a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that (2), (3) and (4) hold. Let Σ be the set of all $\lambda \in \mathbb{R}$ for which the sequence $(e^{-\lambda}A_n)_{n\in\mathbb{Z}}$ does not admit a nonuniform exponential dichotomy on $\mathbb{Z}$ with an arbitrarily small nonuniform part.

Theorem 4.1. *If the dynamics in (60) is Lyapunov regular, then*

$$\Sigma = \{\lambda_1, \dots, \lambda_s\}.$$

Proof. Take $\lambda \in \mathbb{R}$ such that $\lambda \neq \lambda_i$ for $i \in \{1, \dots, s\}$. We note that the Lyapunov exponents associated to the sequence $(e^{-\lambda}A_n)_{n\in\mathbb{Z}}$ are the nonzero numbers $-\lambda+\lambda_i$, for $i = 1, \dots, s$. By Theorem 3 in [7], the sequence $e^{-\lambda}A_n$ admits a nonuniform exponential dichotomy on $\mathbb{Z}_0^+$ with an arbitrarily small nonuniform part, say with projections P_n^+ such that

$$\operatorname{Im} P_0^+ = \bigoplus_{i:\lambda_i<\lambda} E_i.$$

Moreover, the corresponding version of the theorem for $\mathbb{Z}_0^-$ yields that the sequence $e^{-\lambda}A_n$ admits a nonuniform exponential dichotomy on $\mathbb{Z}_0^-$ with an arbitrarily small nonuniform part, say with projections P_n^- such that $\operatorname{Im} P_0^+ = \operatorname{Im} P_0^-$. It follows from Theorem 2.3 that the sequence $e^{-\lambda}A_n$ admits a nonuniform exponential dichotomy on $\mathbb{Z}$ with an arbitrarily small nonuniform part (see (20) and (21)). Thus, $\lambda \notin \Sigma$ and $\Sigma \subset \{\lambda_1, \dots, \lambda_s\}$.

Now we establish the reverse inclusion. Take $i \in \{1, \dots, s\}$ and assume that the sequence $(e^{-\lambda_i}A_n)_{n\in\mathbb{Z}}$ admits a nonuniform exponential dichotomy on $\mathbb{Z}$ with an arbitrarily small nonuniform part. Then there exist projections P_m for $m \in \mathbb{Z}$, a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ satisfying (2) as well as

$$\|\mathcal{A}(m,n)P_n\| \le De^{-(\lambda-\lambda_i)(m-n)+\varepsilon|n|} \quad \text{for} \quad m \ge n \tag{63}$$

and

$$\|\mathcal{A}(m,n)(\operatorname{Id}-P_n)\| \le De^{-(\lambda+\lambda_i)(n-m)+\varepsilon|n|} \quad \text{for} \quad m \le n. \tag{64}$$

Now take $v \in E_i \setminus \{0\}$. By (63), we have

$$\limsup_{m\to+\infty} \frac{1}{m} \log\|\mathcal{A}(m,0)P_0v\| \le -\lambda+\lambda_i < \lambda_i. \tag{65}$$

It follows from (62) and (65) that

$$\limsup_{m\to+\infty} \frac{1}{m} \log\|\mathcal{A}(m,0)(\operatorname{Id}-P_0)v\| \le \lambda_i. \tag{66}$$

On the other hand, by (64),

$$\frac{1}{D}e^{(\lambda+\lambda_i-\varepsilon)m}\|(\operatorname{Id}-P_0)v\| \le \|\mathcal{A}(m,0)(\operatorname{Id}-P_0)v\|$$

for $m \ge 0$. If we would have $P_0v \neq v$, then

$$\limsup_{m\to+\infty} \frac{1}{m} \log\|\mathcal{A}(m,0)(\operatorname{Id}-P_0)v\| \ge \lambda+\lambda_i-\varepsilon > \lambda_i$$

for any sufficiently small $\varepsilon > 0$, which contradicts to (66). Hence, $P_0v = v$. However, by (62) and (65) this is impossible. Therefore, $\lambda_i \in \Sigma$ and since i is arbitrary, we conclude that $\{\lambda_1, \dots, \lambda_s\} \subset \Sigma$. This completes the proof of the theorem. $\square$

4.2. **Continuous time.** Now we consider a linear differential equation

$$x' = A(t)x \tag{67}$$

in $\mathbb{R}^d$, where $A(t)$ is a $d \times d$ matrix varying continuously with $t \in \mathbb{R}$. Let $T(t,s)$ be the evolution family associated to equation (67). Equation (67) is said to be *Lyapunov regular* if there exist a decomposition as in (61) and real numbers $\lambda_1 < \cdots < \lambda_s$ such that:

1. if $i = 1, \dots, s$ and $v \in E_i \setminus \{0\}$, then
$$\lim_{t \to \pm\infty} \frac{1}{t} \log\|T(t,0)v\| = \lambda_i; \tag{68}$$
2. $$\lim_{t \to \pm\infty} \frac{1}{n} \log|\det T(t,0)| = \sum_{i=1}^{s} \lambda_i \dim E_i.$$

We shall say that equation (67) or its evolution family admit a *nonuniform exponential dichotomy on I with an arbitrarily small nonuniform part* if there exist projections P_t for $t \in I$, a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that (37), (38) and (39) hold. For each $\lambda \in \mathbb{R}$ we consider the equation

$$x' = (A(t) - \lambda \mathrm{Id})x$$

and its evolution family

$$T_\lambda(t,s) = e^{-\lambda(t-s)} T(t,s).$$

Let Σ be the set of all $\lambda \in \mathbb{R}$ with the property that the evolution family $T_\lambda(t,s)$ does not admit a nonuniform exponential dichotomy on $\mathbb{R}$ with an arbitrarily small nonuniform part.

Theorem 4.2. *If equation (67) is Lyapunov regular, then*

$$\Sigma = \{\lambda_1, \dots, \lambda_s\}.$$

Proof. The proof is analogous to the proof of Theorem 4.1.

Take $\lambda \in \mathbb{R}$ such that $\lambda \neq \lambda_i$ for $i = 1, \dots, s$. We note that the Lyapunov exponents associated to the evolution family $T_\lambda(t,s)$ are the nonzero numbers $-\lambda + \lambda_i$, for $i = 1, \dots, s$. It follows from Theorem 4 in [6] that $T_\lambda(t,s)$ admits a nonuniform exponential dichotomy on $\mathbb{R}_0^+$ with an arbitrarily small nonuniform part, say with projections P_t^+ such that

$$\operatorname{Im} P_0^+ = \bigoplus_{i:\lambda_i < \lambda} E_i.$$

Moreover, the corresponding version of the theorem for $\mathbb{R}_0^-$ yields that $T_\lambda(t,s)$ admits a nonuniform exponential dichotomy on $\mathbb{R}_0^-$ with an arbitrarily small nonuniform part, say with projections P_t^- such that $\operatorname{Im} P_0^+ = \operatorname{Im} P_0^-$. It follows from Theorem 3.2 that $T_\lambda(t,s)$ admits a nonuniform exponential dichotomy on $\mathbb{R}$ with an arbitrarily small nonuniform part and thus $\lambda \notin \Sigma$. Hence, $\Sigma \subset \{\lambda_1, \dots, \lambda_s\}$.

Now we show that $\{\lambda_1, \dots, \lambda_s\} \subset \Sigma$. Take $i \in \{1, \dots, s\}$ and assume that the evolution family $T_{\lambda_i}(t,s)$ admits a nonuniform exponential dichotomy on $\mathbb{R}$ with an arbitrarily small nonuniform part. Then there exist projections P_t for $t \in \mathbb{R}$, a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ satisfying (37) as well as

$$\|T(t,s)P_s\| \le De^{-(\lambda - \lambda_i)(t-s) + \varepsilon|s|} \quad \text{for} \quad t \ge s \tag{69}$$

and

$$\|T(t,s)(\mathrm{Id} - P_s)\| \le De^{-(\lambda + \lambda_i)(s-t) + \varepsilon|s|} \quad \text{for} \quad t \le s. \tag{70}$$

Now take $v \in E_i \setminus \{0\}$. By (69), we have

$$\limsup_{t\to+\infty} \frac{1}{t} \log\|T(t,0)P_0 v\| \le -\lambda + \lambda_i < \lambda_i. \tag{71}$$

It follows from (68) and (71) that

$$\limsup_{t\to+\infty} \frac{1}{t} \log\|T(t,0)(\mathrm{Id} - P_0)v\| \le \lambda_i.$$

On the other hand, by (70),

$$\|(\mathrm{Id} - P_0)v\| \le De^{-(\lambda+\lambda_i)t+\varepsilon|t|}\|T(t,0)(\mathrm{Id} - P_0)v\|$$

for $t \ge 0$. If we would have $P_0 v \ne v$, then

$$\limsup_{t\to+\infty} \frac{1}{t} \log\|T(t,0)v\| \ge \lambda + \lambda_i - \varepsilon > \lambda_i$$

for any sufficiently small $\varepsilon > 0$, which contradicts to (71). Hence, $P_0 v = v$. However, by (68) and (71) this is impossible. We conclude that $\lambda_i \in \Sigma$ and since i is arbitrary, this completes the proof of the theorem. □

Appendix A. **Proof of Lemma 3.3.** The purpose of this appendix is to prove Lemma 3.3. Incidentally, it is natural to ask whether the spaces Y and Y_1 in the lemma could be the same. In fact, one could replace Y_1 by Y under the additional assumption of bounded growth. This means that there exist $C, d > 0$ such that

$$\|T(t,\tau)x\|_t \le Ce^{d|t-\tau|}\|x\|_\tau$$

for $x \in X$ and $t, \tau \in \mathbb{R}$. However, in general the norms $\|\cdot\|_t$ in (46) need not satisfy this property. We emphasize that the need for bounded growth is not caused by the nonuniform exponential behavior. Indeed, the problem already occurs for a uniform exponential behavior (see [13]), although it does not occur in the case of discrete time.

We proceed with the proof of the lemma. We first observe that:

1. The family of norms $\|\cdot\|_t$ constructed in the proof of Theorem 3.2 satisfies the following property: given $\tau \in \mathbb{R}$, there exists $c > 0$ such that the map $t \mapsto \|T(t,\tau)x\|_t$ is continuous on $[\tau, \tau + c]$ for each $x \in X$. Indeed, for $\tau \ge 0$ we can take an arbitrary $c > 0$, while for $\tau < 0$ we can take any $c > 0$ such that $\tau + c < 0$.
2. Given $\tau \ne 0$, there exists $c > 0$ such that for any map $v\colon [\tau - c, \tau] \to X$ satisfying
$$v(t) = T(t,s)v(s) \quad \text{for} \quad \tau \ge t \ge s \ge \tau - c,$$
the function $s \mapsto \|v(s)\|_s$ is continuous on $[\tau - c, \tau]$. Indeed, for $\tau < 0$ we can take an arbitrary $c > 0$, while for $\tau > 0$ we can take any $c > 0$ such that $\tau - c > 0$. Writing
$$v(t) = T(t, \tau - c)v(\tau - c) \quad \text{for} \quad t \in [\tau - c, \tau],$$
the property follows now from the former observation.

Let R be the linear operator defined by $Rx = y$ in the domain $\mathcal{D}(R)$ formed by all $x \in Y$ for which there exists $y \in Y_1$ such that (47) holds.

Lemma A.1. *The operator $R\colon \mathcal{D}(R) \to Y_1$ is closed.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(R)$ converging to $x \in Y$ such that Rx_n converges to $y \in Y_1$. For each $\tau \in \mathbb{R}$, we have

$$\begin{aligned} x(t) - T(t,\tau)x(\tau) &= \lim_{n\to\infty} (x_n(t) - T(t,\tau)x_n(\tau)) \\ &= \lim_{n\to\infty} \int_\tau^t T(t,s)y_n(s)\,ds \end{aligned}$$

for $t \geq \tau$. Moreover,

$$\begin{aligned} \left\| \int_\tau^t T(t,s)y_n(s)\,ds - \int_\tau^t T(t,s)y(s)\,ds \right\| &\leq M \int_\tau^t \|y_n(s) - y(s)\|\,ds \\ &\leq M \int_\tau^t \|y_n(s) - y(s)\|_s\,ds \\ &\leq M(t-\tau+1)\|y_n - y\|_1, \end{aligned}$$

where

$$M = \sup\left\{\|T(t,s)\| : s \in [\tau,t]\right\} < +\infty$$

(as a consequence of the Banach–Steinhaus theorem). Since y_n converges to y in Y_1, we conclude that

$$\lim_{n\to\infty} \int_\tau^t T(t,s)y_n(s)\,ds = \int_\tau^t T(t,s)y(s)\,ds,$$

and so (12) holds. Hence, $Rx = y$ and $x \in \mathcal{D}(R)$. □

By the closed graph theorem, the operator R has a bounded inverse $G\colon Y_1 \to Y$. For each $\tau \in \mathbb{R}$ let

$$F_\tau^s = \left\{x \in X : \sup_{t\geq\tau}\|T(t,\tau)x\|_t < +\infty\right\}$$

and let F_τ^u be the set of all $x \in X$ for which there exists a continuous function $v\colon (-\infty,\tau] \to X$ with $v(\tau) = x$ such that $\sup_{t\leq\tau}\|v(t)\|_t < +\infty$ and

$$v(t) = T(t,s)v(s) \quad \text{for} \quad \tau \geq t \geq s.$$

Clearly, F_τ^s and F_τ^u are subspaces of X.

Lemma A.2. *For $\tau \in \mathbb{R}$, we have*

$$X = F_\tau^s \oplus F_\tau^u. \tag{72}$$

Proof. Given $x \in X$ and $\tau \in \mathbb{R}$, let

$$g(s) = \chi_{[\tau,\tau+1]}(s)T(s,\tau)x.$$

Clearly, $g \in Y_1$. Since R is invertible, there exists $v \in \mathcal{D}(R)$ such that $Rv = g$. It follows from (47) that

$$v(t) = T(t,\tau)(v(\tau) + x) \quad \text{for} \quad t \geq \tau + 1.$$

Since $v \in Y$, we conclude that $v(\tau) + x \in F_\tau^s$. Similarly, it follows from (47) that $v(t) = T(t,s)v(s)$ for $\tau \geq t \geq s$. Hence, $v(\tau) \in F_\tau^u$ and $x \in F_\tau^s + F_\tau^u$.

Now take $x \in F_\tau^s \cap F_\tau^u$. Then there exists $v\colon (-\infty,\tau] \to X$ continuous with $v(\tau) = x$ such that $\sup_{t\leq\tau}\|v(t)\|_t < +\infty$ and

$$v(t) = T(t,s)v(s) \quad \text{for} \quad \tau \geq t \geq s.$$

We define a map $u\colon \mathbb{R} \to X$ by

$$u(t) = \begin{cases} T(t,\tau)x & \text{if } t \geq \tau, \\ v(t) & \text{if } t \leq \tau. \end{cases}$$

Clearly, u is continuous and $\sup_{t\in\mathbb{R}}\|u(t)\|_t < +\infty$. Moreover, it is easy to verify that

$$u(t) = T(t,s)u(s) \quad \text{for} \quad t \geq s.$$

Hence, $Ru = 0$ and $u \in \mathcal{D}(R)$. Since R is invertible, we conclude that $u = 0$ and so $x = u(\tau) = 0$. ☐

Now let $P(\tau)\colon X \to F_\tau^s$ and $Q(\tau)\colon X \to F_\tau^u$ be the projections associated to the decomposition in (72), with $P(\tau) + Q(\tau) = \mathrm{Id}$. It is easy to verify that

$$P(t)T(t,\tau) = T(t,\tau)P(\tau) \quad \text{for} \quad t \geq \tau.$$

Lemma A.3. *For each $t \geq \tau$, the map $T(t,\tau)|F_\tau^u\colon F_\tau^u \to F_t^u$ is invertible.*

Proof. Assume that $T(t,\tau)x = 0$ for some $x \in F_\tau^u$. Since $x \in F_\tau^u$, there exists a continuous function $v\colon (-\infty,\tau] \to X$ with $v(\tau) = x$ such that $\sup_{s\leq\tau}\|v(s)\|_s < +\infty$ and

$$v(s_1) = T(s_1,s_2)v(s_2) \quad \text{for} \quad \tau \geq s_1 \geq s_2.$$

We define a map $u\colon \mathbb{R} \to X$ by

$$u(s) = \begin{cases} T(s,\tau)x & \text{if } s \geq \tau, \\ v(s) & \text{if } s \leq \tau. \end{cases}$$

Clearly, u is continuous and $\sup_{s\in\mathbb{R}}\|u(s)\|_s < +\infty$ (we note that $u(s) = 0$ for $s \geq t$). Moreover,

$$u(s_1) = T(s_1,s_2)u(s_2) \quad \text{for} \quad s_1 \geq s_2.$$

It follows that $Ru = 0$ and $u \in \mathcal{D}(R)$. Since R is invertible, we obtain $u = 0$ and so $x = u(\tau) = 0$. Therefore, $T(t,\tau)|F_\tau^u\colon F_\tau^u \to F_t^u$ is injective.

Now take $x \in F_t^u$. Then there exists $v\colon (-\infty,t] \to X$ continuous with $v(t) = x$ such that $\sup_{s\leq t}\|v(s)\|_s < +\infty$ and

$$v(s_1) = T(s_1,s_2)v(s_2) \quad \text{for} \quad t \geq s_1 \geq s_2.$$

In particular,

$$x = v(t) = T(t,\tau)v(\tau)$$

and since $v(\tau) \in F_\tau^u$, the map $T(t,\tau)|F_\tau^u\colon F_\tau^u \to F_t^u$ is onto. ☐

Lemma A.4. *There exists $M > 0$ such that*

$$\|P(\tau)x\|_\tau \leq M\|x\|_\tau \tag{73}$$

for $x \in X$ and $\tau \in \mathbb{R}$.

Proof. Given $x \in X$ and $\tau \in \mathbb{R}$, for each $h > 0$ we define $g_h\colon \mathbb{R} \to X$ by

$$g_h(t) = \frac{1}{h}\chi_{[\tau,\tau+h]}(t)T(t,\tau)x.$$

Clearly, $g_h \in Y_1$ and so there exists $v_h \in \mathcal{D}(R)$ such that $Rv_h = g_h$. We have

$$\begin{aligned} \|P(\tau)x\|_\tau &= \|v_h(\tau) + x\|_\tau \leq \|x\|_\tau + \|v_h(\tau)\|_\tau \\ &\leq \|x\|_\tau + \|v_h\|_\infty = \|x\|_\tau + \|Gg_h\|_\infty \end{aligned}$$

(it follows from the proof of Lemma A.2 that $P(\tau)x = v_h(\tau) + x$). Moreover,

$$\|Gg_h\|_\infty \le \|G\| \cdot \|g_h\|_1 \le \|G\| \frac{1}{h} \int_\tau^{\tau+h} \|T(t,\tau)x\|_t \, dt.$$

Letting $h \to 0$, it follows from the observations before Lemma A.1 that

$$\|P(\tau)x\|_\tau \le (1 + \|G\|)\|x\|_\tau$$

and so (73) holds taking $M = 1 + \|G\|$. □

Now we establish bounds along the stable and unstable direction.

Lemma A.5. *There exist constants $\lambda, D > 0$ such that*

$$\|T(t,\tau)P_\tau x\|_t \le De^{-\lambda(t-\tau)}\|x\|_\tau \tag{74}$$

for $x \in X$ and $t \ge \tau$.

Proof. Take $x \in F_\tau^s$ and define a function $u \colon \mathbb{R} \to X$ by

$$u(t) = \chi_{[\tau,\infty)}(t)T(t,\tau)x.$$

Moreover, for each $h > 0$, define a function $\phi_h \colon \mathbb{R} \to \mathbb{R}$ by

$$\phi_h(t) = \begin{cases} 0, & t \le \tau, \\ (t-\tau)/h, & \tau \le t \le \tau + h, \\ 1, & \tau + h \le t. \end{cases}$$

Finally, let

$$g_h(t) = \frac{1}{h}\chi_{[\tau,\tau+h]}(t)T(t,\tau)x.$$

It is easy to verify that $\phi_h u \in \mathcal{D}(R)$, $g_h \in Y_1$ and $R(\phi_h u) = g_h$. Moreover,

$$\begin{aligned} \sup\left\{\|u(t)\|_t : t \in [\tau+h, +\infty)\right\} &= \sup\left\{\|\phi_h(t)u(t)\|_t : t \in [\tau+h, +\infty)\right\} \\ &\le \|\phi_h u\|_\infty = \|Gg_h\|_\infty \le \|G\| \cdot \|g_h\|_1 \\ &\le \|G\| \frac{1}{h} \int_\tau^{\tau+h} \|u(s)\|_s \, ds. \end{aligned}$$

Letting $h \to 0$, it follows from the observations before Lemma A.1 that

$$\|u(t)\|_t \le \|G\| \cdot \|x\|_\tau \quad \text{for} \quad t \ge \tau. \tag{75}$$

We claim that there exists $N \in \mathbb{N}$ such that for every $\tau \in \mathbb{R}$ and $x \in F_\tau^s$,

$$\|u(t)\|_t \le \frac{1}{2}\|x\|_\tau \quad \text{for} \quad t - \tau \ge N, \tag{76}$$

where $u(t) = T(t,\tau)x$. Take $t_0 \in \mathbb{R}$ such that $t_0 > \tau$ and $\|u(t_0)\|_{t_0} > \|x\|_\tau/2$. It follows from (75) that

$$\frac{1}{2\|G\|}\|x\|_\tau < \|u(s)\|_s \le \|G\| \cdot \|x\|_\tau, \quad \tau \le s \le t_0. \tag{77}$$

Now let

$$y(t) = \chi_{[\tau,t_0]}(t)u(t)\|u(t)\|_t^{-1} \quad \text{and} \quad v(t) = u(t)\int_{-\infty}^t \chi_{[\tau,t_0]}(s)\|u(s)\|_s^{-1} \, ds$$

for $t \in \mathbb{R}$. It is easy to verify that $v \in \mathcal{D}(R)$, $y \in Y_1$ and $Rv = y$. Therefore,

$$\|v\|_\infty = \|Gy\|_\infty \le \|G\| \cdot \|y\|_1 \le \|G\|.$$

Hence, it follows from (77) that

$$\|G\| \geq \|v(t_0)\|_{t_0} \geq \|u(t_0)\|_{t_0} \int_\tau^{t_0} \|u(s)\|_s^{-1}\, ds \geq \frac{1}{2\|G\|^2}(t_0-\tau)$$

and so (76) holds taking $N > 2\|G\|^3$.

Now take $t \geq \tau$ and write $t-\tau = kN + r$, with $k \in \mathbb{N}$ and $0 \leq r < N$. By (73), (75) and (76), we obtain

$$\begin{aligned}\|T(t,\tau)P(\tau)x\|_t &= \|T(\tau+kN+r,\tau)P(\tau)x\|_{\tau+kN+r}\\ &\leq \frac{1}{2^k}\|T(\tau+r,\tau)P(\tau)x\|_{\tau+r}\\ &\leq \frac{\|G\|}{2^k}\|P(\tau)x\|_\tau\\ &\leq 2\|G\|Me^{-(t-\tau)\log 2/N}\|x\|_\tau\end{aligned}$$

for $x \in X$. Taking $D = 2M\|G\|$ and $\lambda = \log 2/N$ yields property (74). ☐

Lemma A.6. *There exist constants $\lambda, D > 0$ such that*

$$\|T(t,\tau)Q_\tau x\|_t \leq De^{-\lambda(\tau-t)}\|x\|_\tau \tag{78}$$

for $x \in X$ and $t \leq \tau$.

Proof. We first consider the case when $\tau \neq 0$. Take $x \in F_\tau^u$ and define a function $u\colon \mathbb{R} \to X$ by

$$u(t) = \chi_{(-\infty,\tau]}(t)T(t,\tau)x.$$

Moreover, for each $h > 0$, define a function $\psi_h\colon \mathbb{R} \to \mathbb{R}$ by

$$\psi_h(t) = \begin{cases}1, & t \leq \tau-h,\\ (-t+\tau)/h, & \tau-h \leq t \leq \tau,\\ 0, & \tau \leq t.\end{cases}$$

Finally, let $g_h = -\frac{1}{h}\chi_{[\tau-h,\tau]}u$. It is easy to verify that $\psi_h u \in \mathcal{D}(R)$, $g_h \in Y_1$ and $R(\psi_h u) = g_h$. Moreover,

$$\begin{aligned}\sup\left\{\|u(t)\|_t : t \in (-\infty,\tau-h]\right\} &= \sup\left\{\|\psi_h(t)u(t)\|_t : t \in (-\infty,\tau-h]\right\}\\ &\leq \|\psi_h u\|_\infty = \|Gg_h\|_\infty \leq \|G\|\cdot\|g_h\|_1\\ &\leq \|G\|\frac{1}{h}\int_{\tau-h}^{\tau}\|u(s)\|_s\, ds.\end{aligned}$$

Letting $h \to 0$, it follows from the observations before Lemma A.1 that

$$\|u(t)\|_t \leq \|G\|\cdot\|x\|_\tau \quad \text{for} \quad t \leq \tau. \tag{79}$$

We claim that there exists $N \in \mathbb{N}$ such that for every $\tau \in \mathbb{R}$ and $x \in F_\tau^u$,

$$\|u(t)\|_t \leq \frac{1}{2}\|x\|_\tau \quad \text{for} \quad \tau - t \geq N, \tag{80}$$

Take $t_0 \in \mathbb{R}$ such that $t_0 < \tau$ and $\|u(t_0)\|_{t_0} > \|x\|_\tau/2$. It follows from (79) that

$$\frac{1}{2\|G\|}\|x\|_\tau < \|u(s)\|_s \leq \|G\|\cdot\|x\|_\tau, \quad t_0 \leq s \leq \tau,\ s \neq 0. \tag{81}$$

Now let

$$y(t) = -\chi_{[t_0,\tau]}(t)u(t)\|u(t)\|_t^{-1} \quad \text{and} \quad v(t) = u(t)\int_t^{+\infty}\chi_{[t_0,\tau]}(s)\|u(s)\|_s^{-1}\, ds$$

for $t \in \mathbb{R}$. It is easy to verify that $v \in \mathcal{D}(R)$, $y \in Y_1$ and $Rv = y$. Therefore,

$$\|v\|_\infty = \|Gy\|_\infty \leq \|G\| \cdot \|y\|_1 \leq \|G\|.$$

Hence, it follows from (81) that

$$\|G\| \geq \|v(t_0)\|_{t_0} \geq \|u(t_0)\|_{t_0} \int_{t_0}^{\tau} \|u(s)\|_s^{-1}\, ds \geq \frac{1}{2\|G\|^2}(\tau - t_0)$$

and so (80) holds taking $N > 2\|G\|^3$.

Now take $t \leq \tau$ and write $\tau - t = kN + r$, with $k \in \mathbb{N}$ and $0 \leq r < N$. By (73), (79) and (80), we obtain

$$\begin{aligned}
\|T(t,\tau)Q(\tau)x\|_t &= \|T(\tau - kN - r, \tau)Q(\tau)x\|_{\tau - kN - r} \\
&\leq \frac{1}{2^k}\|T(\tau - r, \tau)Q(\tau)x\|_{\tau - r} \\
&\leq \frac{\|G\|}{2^k}\|Q(\tau)x\|_\tau \\
&\leq 2\|G\|(1+M)e^{-(\tau - t)\log 2/N}\|x\|_\tau
\end{aligned}$$

for $x \in X$. Taking $D = 2(1+M)\|G\|$ and $\lambda = \log 2/N$ yields property (78).

Finally, we consider the case when $\tau = 0$. Take $x \in F_0^u$. For each $n \in \mathbb{N}$ and $t \leq 0$, we have

$$\begin{aligned}
\|T(t,0)x\|_t &= \|T(t,1/n)T(1/n,0)x\|_t \\
&\leq De^{-\lambda(1/n - t)}\|T(1/n,0)x\|_{1/n}.
\end{aligned} \tag{82}$$

Since $\|T(1/n,0)x\|_{1/n} \to \|x\|_0$ when $n \to \infty$, letting $n \to \infty$ in (82) yields that

$$\|T(t,0)x\|_t \leq De^{\lambda t}\|x\|_0 \quad \text{for} \quad x \in \operatorname{Im} Q_0,\ t \leq 0.$$

This shows that (78) also holds for $\tau = 0$. $\square$

Therefore, $T(t,\tau)$ admits a nonuniform exponential dichotomy with respect to the family of norms $\|\cdot\|_t$ with $\varepsilon = 0$.

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