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Admissibility and Nonuniform Exponential Trichotomies

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Abstract—For a nonautonomous dynamics defined by a sequence of linear operators acting on a Banach space, we show that the notion of a nonuniform exponential trichotomy can be completely characterized in terms of admissibility properties. This refers to the existence of bounded solutions under any bounded time-dependent perturbation of certain homotheties of the original dynamics. We also consider the more restrictive notion of a strong nonuniform exponential trichotomy and again we give a characterization in terms of admissibility properties. We emphasize that both notions are ubiquitous in the context of ergodic theory. As a nontrivial application, we show in a simple manner that the two notions of trichotomy persist under sufficiently small linear perturbations. Finally, we obtain a corresponding characterization of nonuniformly partially hyperbolic sets.

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1. INTRODUCTION

Our main objective is to give a complete characterization of several variations of the notion of an *exponential trichotomy* in terms of admissibility properties. More precisely, we look at the following three general situations:

1. We consider the general case of a *nonuniform* exponential trichotomy. This means that we allow a nonuniform (conditional) exponential stability on the initial time. For example, almost all orbits of a measure-preserving flow and so, in particular, of any Hamiltonian flow on a compact energy level have this behavior.
2. We also consider the notion of a *strong* nonuniform exponential trichotomy and give a corresponding characterization. This means that there are both lower and upper exponential bounds on the stable and unstable directions, instead of only on the central direction. We note that this is again a common behavior in the context of ergodic theory.
3. Finally, we obtain a corresponding characterization of the notion of a nonuniformly partially hyperbolic set. This corresponds to considering various trajectories simultaneously instead of a single one. For that we profit from having already given a characterization of the notion of a nonuniformly hyperbolic set in terms of an admissibility property.

As an application of our results, we give short proofs of the robustness of a nonuniform exponential trichotomy and of a strong nonuniform exponential trichotomy.

The study of admissibility properties goes back to the pioneering work of Perron in [11] and referred originally to the existence of bounded solutions of the equation

$$x' = A(t)x + f(t)$$

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in a finite-dimensional space $\mathbb{R}^n$ for any bounded continuous perturbation f. In particular, he showed that such properties can be used to deduce the stability or the conditional stability under sufficiently small perturbations. We note that the study of the conditional stability is naturally related to the existence of stable and unstable manifolds. A relatively simple modification of Perron's work for continuous time yields the following result for discrete time.

Theorem 1. *Let $(A_m)_{m\in\mathbb{N}}$ be a sequence of $n \times n$ matrices. If for each bounded sequence $(f_m)_{m\in\mathbb{N}} \subset \mathbb{R}^n$ there exists $x_0 \in \mathbb{R}^n$ such that the sequence*

$$x_m = A_{m-1}x_{m-1} + f_m, \quad m \in \mathbb{N} \tag{1.1}$$

is bounded, then any bounded sequence $(A_m \cdots A_1 x)_{m\in\mathbb{N}}$ tends to zero as $m \to \infty$.

Theorem 1 can be rephrased by saying that an admissibility property, in this case requiring that for any bounded perturbation $(f_m)_{m\in\mathbb{N}}$ the solution $(x_m)_{m\in\mathbb{N}}$ of system (1.1) is bounded, implies that the linear dynamics is Lyapunov stable. There is an extensive literature on the relation between admissibility properties and stability properties. In particular, one can consider different spaces in which we look for the perturbation $(f_m)_{m\in\mathbb{N}}$ and the solution $(x_m)_{m\in\mathbb{N}}$. Moreover, if the solutions belong to certain smaller spaces, say with some particular decay at infinity, then one can obtain information about the speed of decay of the original linear dynamics. For some of the most relevant early contributions in the area we refer to the books by Massera and Schäffer [9] and by Dalec′kiĭ and Kreĭn [5]. See [8] for some early results in infinite-dimensional spaces. For a detailed list of references we refer to [4, 7].

As detailed above, one of the objectives of our paper is to obtain a characterization of the notion of a nonuniformly partially hyperbolic set in terms of admissibility properties. This notion arises naturally in the context of smooth ergodic theory. Indeed, if f is a C^1 diffeomorphism preserving a finite measure μ, then there exists a nonuniformly partially hyperbolic set of full μ-measure (see [3]). Our work is close in spirit to that of Mather in [10], who obtained a similar characterization for uniformly hyperbolic sets, as well as to that of Dragičević and Slijepčević [6], where the problem of extending Mather's result to nonuniformly hyperbolic dynamics was considered for the first time. The present paper is the first to deal with a nonuniformly partially hyperbolic dynamics.

2. PRELIMINARIES

Let $X = (X, \|\cdot\|)$ be a Banach space and let $B(X)$ be the set of all bounded linear operators acting on X. Given a sequence $(A_m)_{m\in\mathbb{Z}}$ of invertible linear operators in $B(X)$, we define

$$\mathcal{A}(n,m) = \begin{cases} A_{n-1}\cdots A_m & \text{if } n > m, \\ \mathrm{Id} & \text{if } n = m, \\ A_n^{-1}\cdots A_{m-1}^{-1} & \text{if } n < m. \end{cases}$$

We say that $(A_m)_{m\in\mathbb{Z}}$ admits a *nonuniform exponential trichotomy* if there exist projections $P_m^i\colon X \to X$ for $i \in \{1,2,3\}$ and $m \in \mathbb{Z}$ satisfying

$$P_m^1 + P_m^2 + P_m^3 = \mathrm{Id}, \quad A_m P_m^i = P_{m+1}^i A_m$$

for $m \in \mathbb{Z}$ and $i \in \{1,2,3\}$, and there exist constants

$$D > 0, \quad 0 \leqslant a < b, \quad 0 \leqslant c < d, \quad \varepsilon \geqslant 0 \tag{2.1}$$

such that

$$\|\mathcal{A}(m,n)P_n^1\| \leqslant De^{-d(m-n)+\varepsilon|n|}, \quad \|\mathcal{A}(m,n)P_n^3\| \leqslant De^{a(m-n)+\varepsilon|n|} \tag{2.2}$$

for $m, n \in \mathbb{Z}$ with $m \geqslant n$ and

$$\|\mathcal{A}(m,n)P_n^2\| \leqslant De^{-b(n-m)+\varepsilon|n|}, \quad \|\mathcal{A}(m,n)P_n^3\| \leqslant De^{c(n-m)+\varepsilon|n|} \tag{2.3}$$

for $m, n \in \mathbb{Z}$ with $m \leqslant n$. Moreover, we say that the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a *nonuniform exponential dichotomy* if it admits a nonuniform exponential trichotomy with $P_m^3 = 0$ for $m \in \mathbb{Z}$.

We also recall the notion of an exponential dichotomy with respect to a sequence of norms. Let $\|\cdot\|_m$, for $m \in \mathbb{Z}$, be a sequence of norms on X such that $\|\cdot\|_m$ is equivalent to $\|\cdot\|$ for each m. We say that $(A_m)_{m\in\mathbb{Z}}$ admits an *exponential dichotomy* with respect to the sequence of norms $\|\cdot\|_m$ if there exist projections $P_m \colon X \to X$ for $m \in \mathbb{Z}$ satisfying

$$A_m P_m = P_{m+1} A_m \quad \text{for} \quad m \in \mathbb{Z},$$

and there exist constants $\lambda, D > 0$ such that for each $x \in X$ and $n, m \in \mathbb{Z}$ we have

$$\|\mathcal{A}(n,m)P_m x\|_n \leqslant De^{-\lambda(n-m)}\|x\|_m \quad \text{for} \quad n \geqslant m \tag{2.4}$$

and

$$\|\mathcal{A}(n,m)Q_m x\|_n \leqslant De^{-\lambda(m-n)}\|x\|_m \quad \text{for} \quad n \leqslant m, \tag{2.5}$$

where $Q_m = \operatorname{Id} - P_m$. The following auxiliary result gives a characterization of the spaces $\operatorname{Im} P_n$ and $\operatorname{Im} Q_n$.

Proposition 1. *For each $n \in \mathbb{Z}$, we have*

$$\operatorname{Im} P_n = \left\{ x \in X : \sup_{m \geqslant n} \|\mathcal{A}(m,n)x\|_m < +\infty \right\}$$

and

$$\operatorname{Im} Q_n = \left\{ x \in X : \sup_{m \leqslant n} \|\mathcal{A}(m,n)x\|_m < +\infty \right\}.$$

Proof. It follows readily from (2.4) that

$$\sup_{m \geqslant n} \|\mathcal{A}(m,n)x\|_m < +\infty \tag{2.6}$$

for $x \in \operatorname{Im} P_n$. Now take $x \in X$ satisfying (2.6). Since $x = P_n x + Q_n x$, it follows from (2.4) that

$$\sup_{m \geqslant n} \|\mathcal{A}(m,n)Q_n x\|_m < +\infty.$$

On the other hand, by (2.5), we have

$$\|Q_n x\|_n = \|\mathcal{A}(n,m)\mathcal{A}(m,n)Q_n x\|_n \leqslant e^{-\lambda(m-n)}\|\mathcal{A}(m,n)Q_n x\|_m$$

for $m \geqslant n$. Letting $m \to \infty$, we obtain $Q_n x = 0$ and thus $x = P_n x \in \operatorname{Im} P_n$. This establishes the first identity in the proposition. The second identity can be obtained in a similar manner. □

The following result taken from [1] establishes the connection between the notions of a nonuniform exponential dichotomy and an exponential dichotomy with respect to a sequence of norms.

Proposition 2. *The following properties are equivalent:*

1. *$(A_m)_{m\in\mathbb{Z}}$ admits a nonuniform exponential dichotomy;*
2. *$(A_m)_{m\in\mathbb{Z}}$ admits an exponential dichotomy with respect to a sequence of norms $\|\cdot\|_m$ satisfying*

$$\|x\| \leqslant \|x\|_n \leqslant De^{\varepsilon|n|}\|x\|, \quad n \in \mathbb{Z}, \ x \in X \tag{2.7}$$

for some constant $D > 0$.

We note that the constant ε is the same in both properties (more precisely, in inequalities (2.2)–(2.3) and in property (2.7)).

3. EXPONENTIAL TRICHOTOMIES AND ADMISSIBILITY

In this section we characterize the notion of a nonuniform exponential trichotomy via admissibility properties.

First we recall the concept of admissibility with respect to a sequence of norms $\|\cdot\|_m$. Let Y be the set of all sequences $\mathbf{x} = (x_m)_{m\in\mathbb{Z}}$ with $x_m \in X$ for $m \in \mathbb{Z}$ such that

$$\|\mathbf{x}\|_\infty := \sup_{m\in\mathbb{Z}} \|x_m\|_m < +\infty.$$

It is easy to verify that $Y = (Y, \|\cdot\|_\infty)$ is a Banach space. We say that a sequence of linear operators $(A_m)_{m\in\mathbb{Z}}$ has an *admissibility property* with respect to the sequence of norms $\|\cdot\|_m$ if for each $\mathbf{y} = (y_m)_{m\in\mathbb{Z}} \in Y$ there exists a unique $\mathbf{x} = (x_m)_{m\in\mathbb{Z}} \in Y$ such that

$$x_m - A_{m-1}x_{m-1} = y_m \quad \text{for} \quad m \in \mathbb{Z}.$$

The following is our first main result.

Theorem 2. *Assume that the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a nonuniform exponential trichotomy with $\varepsilon < b + d$. Then there exist sequences of norms $\|\cdot\|_{1,m}$ and $\|\cdot\|_{2,m}$ for $m \in \mathbb{Z}$ and constants $D', \omega > 0$ and $\omega' < 0$ with $\varepsilon \leqslant \omega - \omega'$ such that:*

1. *$(e^{\omega}A_m)_{m\in\mathbb{Z}}$ has an admissibility property with respect to the sequence of norms $\|\cdot\|_{1,m}$;*

2. *$(e^{\omega'}A_m)_{m\in\mathbb{Z}}$ has an admissibility property with respect to the sequence of norms $\|\cdot\|_{2,m}$;*

3. *for $m \in \mathbb{Z}$, $i \in \{1, 2\}$ and $x \in X$, we have*

$$\|x\| \leqslant \|x\|_{i,m} \leqslant D'e^{\varepsilon|m|}\|x\|. \tag{3.1}$$

Proof. Take $\omega \in (c, d)$ and consider the sequence $B_m = e^{\omega}A_m$. Then

$$\mathcal{B}(m, n) = e^{\omega(m-n)}\mathcal{A}(m, n)$$

and it follows from (2.2) and (2.3) that

$$\|\mathcal{B}(m, n)P_n^1\| \leqslant De^{-(d-\omega)(m-n)+\varepsilon|n|} \tag{3.2}$$

for $m \geqslant n$ and that

$$\|\mathcal{B}(m, n)P_n^2\| \leqslant De^{-(b+\omega)(n-m)+\varepsilon|n|} \tag{3.3}$$

and

$$\|\mathcal{B}(m, n)P_n^3\| \leqslant De^{-(\omega-c)(n-m)+\varepsilon|n|} \tag{3.4}$$

for $m \leqslant n$. This shows that the sequence $(B_m)_{m\in\mathbb{Z}}$ admits a nonuniform exponential dichotomy with projections $P_m = P_m^1$. By Proposition 2, it admits an exponential dichotomy with respect to a sequence of norms $\|\cdot\|_{1,m}$ satisfying (3.1) for some $D' > 0$.

Now we use the following result established in [1].

Lemma 1. *The following statements are equivalent:*

1. *$(A_m)_{m\in\mathbb{Z}}$ admits an exponential dichotomy with respect to the sequence of norms $\|\cdot\|_m$;*

2. *$(A_m)_{m\in\mathbb{Z}}$ has an admissibility property with respect to a sequence of norms $\|\cdot\|_m$.*

It follows from Lemma 1 that the sequence $(B_m)_{m\in\mathbb{Z}}$ has an admissibility property with respect to the sequence of norms $\|\cdot\|_{1,m}$.

Now take $\omega' \in (-b,-a)$ and consider the sequence $B'_m = e^{\omega'} A_m$. Then

$$\mathcal{B}'(m,n) = e^{\omega'(m-n)}\mathcal{A}(m,n)$$

and it follows from (2.2) and (2.3) that

$$\|\mathcal{B}'(m,n)P^1_n\| \leqslant De^{-(d-\omega')(m-n)+\varepsilon|n|} \tag{3.5}$$

and

$$\|\mathcal{B}'(m,n)P^3_n\| \leqslant De^{-(-a-\omega')(m-n)+\varepsilon|n|} \tag{3.6}$$

for $m \geqslant n$ and that

$$\|\mathcal{B}'(m,n)P^2_n\| \leqslant De^{-(b+\omega')(n-m)+\varepsilon|n|} \tag{3.7}$$

for $m \leqslant n$. This shows that the sequence $(B'_m)_{m\in\mathbb{Z}}$ admits a nonuniform exponential dichotomy with projections $P_m = P^1_m + P^3_m$. Hence, it follows from Proposition 2 and Lemma 1 that it has an admissibility property with respect to a sequence of norms $\|\cdot\|_{2,m}$ satisfying (3.1) for some $D' > 0$.

Finally, since $\varepsilon < b + d$, one can choose ω and ω' so that $\varepsilon \leqslant \omega - \omega'$. $\square$

Now we establish the converse of Theorem 2.

Theorem 3. *Assume that there exist sequences of norms $\|\cdot\|_{1,m}$ and $\|\cdot\|_{2,m}$ for $m \in \mathbb{Z}$ and constants $D', \omega > 0$, $\varepsilon \geqslant 0$ and $\omega' < 0$ with $\varepsilon \leqslant \omega - \omega'$ satisfying properties* 1–3 *of Theorem* 2*. Then the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a nonuniform exponential trichotomy.*

Proof. It follows from Lemma 1 that the sequences $B_m = e^{\omega}A_m$ and $B'_m = e^{\omega'}A_m$ admit exponential dichotomies, respectively, with respect to some sequences of norms $\|\cdot\|_{1,m}$ and $\|\cdot\|_{2,m}$. Hence, there exist projections P^1_m and P^2_m for $m \in \mathbb{Z}$ satisfying

$$B_m P^1_m = P^1_{m+1} B_m, \quad B'_m P^2_m = P^2_{m+1} B'_m$$

for $m \in \mathbb{Z}$ and there exist constants $\lambda, D > 0$ such that for each $x \in X$ and $n, m \in \mathbb{Z}$ we have

$$\|\mathcal{B}(m,n)P^1_n x\|_{1,m} \leqslant De^{-\lambda(m-n)}\|x\|_{1,n}, \tag{3.8}$$

$$\|\mathcal{B}'(m,n)P^2_n x\|_{2,m} \leqslant De^{-\lambda(m-n)}\|x\|_{2,n} \tag{3.9}$$

for $m \geqslant n$ and

$$\|\mathcal{B}(m,n)Q^1_n x\|_{1,m} \leqslant De^{-\lambda(n-m)}\|x\|_{1,n}, \tag{3.10}$$

$$\|\mathcal{B}'(m,n)Q^2_n x\|_{2,m} \leqslant De^{-\lambda(n-m)}\|x\|_{2,n} \tag{3.11}$$

for $m \leqslant n$, where $Q^i_n = \mathrm{Id} - P^i_n$.

Lemma 2. *For each $n \in \mathbb{Z}$, we have*

$$\operatorname{Im} P^1_n \subset \operatorname{Im} P^2_n \quad \textit{and} \quad \operatorname{Im} Q^2_n \subset \operatorname{Im} Q^1_n. \tag{3.12}$$

Proof (of the lemma). Take $x \in \operatorname{Im} P^1_n$. By (3.1), we have

$$\begin{aligned}
\|\mathcal{B}'(m,n)x\|_{2,m} &= e^{\omega'(m-n)}\|\mathcal{A}(m,n)x\|_{2,m} \\
&\leqslant D'e^{\omega'(m-n)}e^{\varepsilon|m|}\|\mathcal{A}(m,n)x\| \\
&\leqslant D'e^{\omega'(m-n)}e^{\varepsilon|m|}\|\mathcal{A}(m,n)x\|_{1,m} \\
&= D'e^{(\omega'-\omega)(m-n)}e^{\varepsilon|m|}\|\mathcal{B}(m,n)x\|_{1,m}
\end{aligned}$$

for $m \geqslant n$. Since $\varepsilon \leqslant \omega - \omega'$, it follows from Proposition 1 that

$$\sup_{m \geqslant n} \|\mathcal{B}'(m,n)x\|_{2,m} < +\infty$$

and hence $x \in \operatorname{Im} P_n^2$ (again from Proposition 1). The proof of the second inclusion in (3.12) is analogous. □

Lemma 3. *The map* $\operatorname{Id} - P_n^1 - Q_n^2$ *is a projection for each* $n \in \mathbb{Z}$.

Proof (of the lemma). It follows from the previous lemma that

$$P_n^1 Q_n^2 = Q_n^2 P_n^1 = 0$$

for $n \in \mathbb{Z}$. Hence

$$\begin{aligned}(\operatorname{Id} - P_n^1 - Q_n^2)^2 &= \operatorname{Id} - 2P_n^1 - 2Q_n^2 + (P_n^1)^2 + (Q_n^2)^2 + P_n^1 Q_n^2 + Q_n^2 P_n^1 \\ &= \operatorname{Id} - P_n^1 - Q_n^2\end{aligned}$$

and the conclusion in the lemma follows. □

Lemma 4. *For each* $n \in \mathbb{Z}$, *we have*

$$\operatorname{Im}(\operatorname{Id} - P_n^1 - Q_n^2) = \operatorname{Im} P_n^2 \cap \operatorname{Im} Q_n^1.$$

Proof (of the lemma). Take $x \in \operatorname{Im} P_n^2 \cap \operatorname{Im} Q_n^1$. We have $Q_n^2 x = P_n^1 x = 0$ and thus,

$$(\operatorname{Id} - P_n^1 - Q_n^2)x = x.$$

This implies that $x \in \operatorname{Im}(\operatorname{Id} - P_n^1 - Q_n^2)$. Now take $x \in \operatorname{Im}(\operatorname{Id} - P_n^1 - Q_n^2)$. We have $P_n^1 x = -Q_n^2 x$. Applying P_n^1, we obtain $P_n^1 x = 0$ and thus $x \in \operatorname{Im} Q_n^1$. Similarly, $x \in \operatorname{Im} P_n^2$ and so $x \in \operatorname{Im} P_n^2 \cap \operatorname{Im} Q_n^1$. □

We proceed with the proof of the theorem. It follows from (3.1) and (3.8) that

$$\|\mathcal{A}(m,n)P_n^1\| \leqslant DD' e^{-(\lambda+\omega)(m-n)+\varepsilon|n|} \quad \text{for} \quad m \geqslant n. \tag{3.13}$$

Similarly, by (3.1) and (3.11) we have

$$\|\mathcal{A}(m,n)Q_n^2\| \leqslant DD' e^{-(\lambda-\omega')(n-m)+\varepsilon|n|} \quad \text{for} \quad m \leqslant n. \tag{3.14}$$

Moreover, it follows from (3.1), (3.9) and (3.10) together with Lemma 4 that for each $x \in \operatorname{Im}(\operatorname{Id} - P_n^1 - Q_n^2)$, we have

$$\|\mathcal{A}(m,n)x\| \leqslant DD' e^{-(\lambda+\omega')(m-n)+\varepsilon|n|}$$

for $m \geqslant n$ and

$$\|\mathcal{A}(m,n)x\| \leqslant DD' e^{-(\lambda-\omega)(n-m)+\varepsilon|n|}$$

for $m \leqslant n$. In addition, by (3.1), (3.8) and (3.11),

$$\|\operatorname{Id} - P_n^1 - Q_n^2\| \leqslant 3DD' e^{\varepsilon|n|}$$

for $n \in \mathbb{Z}$. Hence,

$$\|\mathcal{A}(m,n)(\operatorname{Id} - P_n^1 - Q_n^2)\| \leqslant 3(DD')^2 e^{-(\lambda+\omega')(m-n)+2\varepsilon|n|} \quad \text{for} \quad m \geqslant n \tag{3.15}$$

and

$$\|\mathcal{A}(m,n)(\operatorname{Id} - P_n^1 - Q_n^2)\| \leqslant 3(DD')^2 e^{-(\lambda-\omega)(n-m)+2\varepsilon|n|} \quad \text{for} \quad m \leqslant n. \tag{3.16}$$

It follows from (3.13), (3.14), (3.15) and (3.16) that the sequence $(A_m)_{m \in \mathbb{Z}}$ admits a nonuniform exponential trichotomy. □

4. STRONG NONUNIFORM EXPONENTIAL TRICHOTOMIES

In this section we consider the more restrictive notion of a strong nonuniform exponential trichotomy and we also characterize it via admissibility properties.

We say that a sequence $(A_m)_{m\in\mathbb{Z}}$ admits a *strong nonuniform exponential trichotomy* if it admits a nonuniform exponential trichotomy and there exist constants $d' \geqslant d$ and $b' \geqslant b$ (see (2.1)) such that

$$\|\mathcal{A}(m,n)P_n^1\| \leqslant De^{d'(n-m)+\varepsilon|n|} \quad \text{for} \quad m \leqslant n$$

and

$$\|\mathcal{A}(m,n)P_n^2\| \leqslant De^{b'(m-n)+\varepsilon|n|} \quad \text{for} \quad m \geqslant n.$$

The following is a version of Theorem 2 for the notion of a strong nonuniform exponential trichotomy.

Theorem 4. *Assume that the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a strong nonuniform exponential trichotomy with $\varepsilon < b+d$. Then there exist sequences of norms $\|\cdot\|_{1,m}$ and $\|\cdot\|_{2,m}$ for $m\in\mathbb{Z}$ and constants $D',\omega>0$, $C_1,C_2>1$ and $\omega'<0$ with $\varepsilon\leqslant\omega-\omega'$ satisfying properties 1–3 of Theorem 2 and such that*

$$\frac{1}{C_1}\|x\|_{1,n} \leqslant \|A_n x\|_{1,n+1} \quad \text{and} \quad \|A_n x\|_{2,n+1} \leqslant C_2\|x\|_{2,n} \tag{4.1}$$

for $n\in\mathbb{Z}$ and $x\in X$.

Proof. Take $\omega\in(c,d)$ and consider the sequence $B_m = e^{\omega}A_m$. In addition to (3.2), (3.3) and (3.4), we have

$$\|\mathcal{B}(m,n)P_n^1\| \leqslant De^{(d'-\omega)(n-m)+\varepsilon|n|} \quad \text{for} \quad m \leqslant n. \tag{4.2}$$

Now we introduce new norms. For $n\in\mathbb{Z}$ and $x\in X$, let

$$\begin{aligned}\|x\|_{1,n} = {} & \sup_{m\geqslant n}\big(\|\mathcal{B}(m,n)P_n^1x\|e^{\lambda(m-n)}\big)\\ & + \sup_{m\leqslant n}\big(\|\mathcal{B}(m,n)(\mathrm{Id}-P_n^1)x\|e^{\lambda(n-m)}\big)\\ & + \sup_{m<n}\big(\|\mathcal{B}(m,n)P_n^1x\|e^{-(d'-\omega)(n-m)}\big),\end{aligned}$$

where

$$\lambda = \min\big\{d-\omega, b+\omega, \omega-c\big\} > 0.$$

It follows from (3.2), (3.3), (3.4) and (4.2) that the norms $\|\cdot\|_{1,m}$ satisfy property (3.1) with $D' = 4D$.

Lemma 5. *The sequence $(e^{\omega}A_m)_{m\in\mathbb{Z}}$ admits an exponential dichotomy with respect to the sequence of norms $\|\cdot\|_{1,m}$. Moreover, there exists $C_1>0$ such that the first inequality in (4.1) holds for $n\in\mathbb{Z}$ and $x\in X$.*

Proof (of the lemma). For $m\geqslant n$, since $\lambda < d'-\omega$ we have

$$\begin{aligned}\|\mathcal{B}(m,n)P_n^1x\|_{1,m} &= \sup_{k\geqslant m}\big(\|\mathcal{B}(k,n)P_n^1x\|e^{\lambda(k-m)}\big) + \sup_{k<m}\big(\|\mathcal{B}(k,n)P_n^1x\|e^{-(d'-\omega)(m-k)}\big)\\ &\leqslant \sup_{k\geqslant m}\big(\|\mathcal{B}(k,n)P_n^1x\|e^{\lambda(k-m)}\big)\\ &\quad + \sup_{n\leqslant k<m}\big(\|\mathcal{B}(k,n)P_n^1x\|e^{-\lambda(m-k)}\big)\\ &\quad + \sup_{k<n}\big(\|\mathcal{B}(k,n)P_n^1x\|e^{-(d'-\omega)(m-k)}\big)\\ &\leqslant 2e^{-\lambda(m-n)}\sup_{k\geqslant n}\big(\|\mathcal{B}(k,n)P_n^1x\|e^{\lambda(k-n)}\big)\\ &\quad + e^{-(d'-\omega)(m-n)}\sup_{k<n}\big(\|\mathcal{B}(k,n)P_n^1x\|e^{-(d'-\omega)(n-k)}\big)\\ &\leqslant 2e^{-\lambda(m-n)}\|x\|_{1,n}.\end{aligned} \tag{4.3}$$

One can show in a similar manner that

$$\|\mathcal{B}(m,n)P_n^1 x\|_{1,m} \leqslant 2e^{(d'-\omega)(n-m)}\|x\|_{1,n} \quad \text{for} \quad m \leqslant n. \tag{4.4}$$

Moreover,

$$\begin{aligned}\|\mathcal{B}(m,n)(\mathrm{Id}-P_n^1)x\|_{1,m} &= \sup_{k\leqslant m}\big(\|\mathcal{B}(k,m)\mathcal{B}(m,n)(\mathrm{Id}-P_n^1)x\|e^{\lambda(m-k)}\big)\\ &= e^{-\lambda(n-m)}\sup_{k\leqslant m}\big(\|\mathcal{B}(k,n)(\mathrm{Id}-P_n^1)x\|e^{\lambda(n-k)}\big)\\ &\leqslant e^{-\lambda(n-m)}\|x\|_{1,n}\end{aligned} \tag{4.5}$$

for $m \leqslant n$. It follows from (4.3) and (4.5) that the sequence $(e^{\omega}A_m)_{m\in\mathbb{Z}}$ admits an exponential dichotomy with respect to the sequence of norms $\|\cdot\|_{1,m}$. By (4.4) and (4.5), we have

$$\begin{aligned}e^{-\omega}\|A_n^{-1}x\|_{1,n} &= \|\mathcal{B}(n,n+1)x\|_{1,n}\\ &\leqslant \|\mathcal{B}(n,n+1)P_{n+1}^1 x\|_{1,n} + \|\mathcal{B}(n,n+1)(\mathrm{Id}-P_{n+1}^1)x\|_{1,n}\\ &\leqslant 2e^{d'-\omega}\|x\|_{1,n+1} + e^{-\lambda}\|x\|_{1,n+1}\\ &\leqslant 3e^{d'-\omega}\|x\|_{1,n+1}\end{aligned}$$

for $x \in X$ and $n \in \mathbb{Z}$. This shows that the first inequality in (4.1) holds with $C_1 = 3e^{d'}$. □

By Lemmas 1 and 5, the sequence $(e^{\omega}A_m)_{m\in\mathbb{Z}}$ has an admissibility property with respect to the sequence of norms $\|\cdot\|_{1,m}$.

Now take $\omega' \in (-b,-a)$ and consider the sequence $B'_m = e^{\omega'}A_m$. In addition to (3.5), (3.6) and (3.7), we have

$$\|\mathcal{B}'(m,n)P_n^2\| \leqslant De^{(b'+\omega')(m-n)+\varepsilon|n|} \quad \text{for} \quad m \geqslant n. \tag{4.6}$$

For $n \in \mathbb{Z}$ and $x \in X$, let

$$\begin{aligned}\|x\|_{2,n} = &\sup_{m\geqslant n}\big(\|\mathcal{B}'(m,n)(\mathrm{Id}-P_n^2)x\|e^{\lambda'(m-n)}\big)\\ &+\sup_{m\leqslant n}\big(\|\mathcal{B}'(m,n)P_n^2 x\|e^{\lambda'(n-m)}\big)\\ &+\sup_{m>n}\big(\|\mathcal{B}'(m,n)P_n^2 x\|e^{-(b'+\omega')(m-n)}\big),\end{aligned}$$

where

$$\lambda' = \min\{d-\omega', -a-\omega', b+\omega'\} > 0.$$

It follows from (3.5), (3.6) and (3.7) and (4.6) that the norms $\|\cdot\|_{2,m}$ satisfy property (3.1) with $D' = 4D$.

Lemma 6. *The sequence $(e^{\omega'}A_m)_{m\in\mathbb{Z}}$ admits an exponential dichotomy with respect to the sequence of norms $\|\cdot\|_{2,m}$. Moreover, there exists $C_2 > 0$ such that the second inequality in (4.1) holds for $n \in \mathbb{Z}$ and $x \in X$.*

Proof (of the lemma). For $m \leqslant n$, since $\lambda' < b' + \omega'$ we have

$$\begin{aligned}\|\mathcal{B}'(m,n)P_n^2 x\|_{2,m} &= \sup_{k\leqslant m}\big(\|\mathcal{B}'(k,n)P_n^2 x\|e^{\lambda'(m-k)}\big) + \sup_{k>m}\big(\|\mathcal{B}'(k,n)P_n^2 x\|e^{-(b'+\omega')(k-m)}\big)\\ &\leqslant \sup_{k\leqslant m}\big(\|\mathcal{B}'(k,n)P_n^2 x\|e^{\lambda'(m-k)}\big) + \sup_{n\geqslant k>m}\big(\|\mathcal{B}'(k,n)P_n^2 x\|e^{\lambda'(m-k)}\big)\\ &\quad+\sup_{k>n}\big(\|\mathcal{B}'(k,n)P_n^2 x\|e^{-(b'+\omega')(k-m)}\big)\\ &\leqslant 2e^{-\lambda'(n-m)}\sup_{k\leqslant n}\big(\|\mathcal{B}'(k,n)P_n^2 x\|e^{\lambda'(n-k)}\big)\\ &\quad+e^{-(b'+\omega')(n-m)}\sup_{k>n}\big(\|\mathcal{B}'(k,n)P_n^2 x\|e^{-(b'+\omega')(k-n)}\big)\\ &\leqslant 2e^{-\lambda'(n-m)}\|x\|_{2,n}.\end{aligned} \tag{4.7}$$

One can show in a similar manner that

$$\|\mathcal{B}'(m,n)P_n^2x\|_{2,m} \leqslant 2e^{(b'+\omega')(m-n)}\|x\|_{2,n} \tag{4.8}$$

and

$$\|\mathcal{B}'(m,n)(\mathrm{Id}-P_n^2)x\|_{2,m} \leqslant e^{-\lambda'(m-n)}\|x\|_{2,n} \tag{4.9}$$

for $m \geqslant n$. It follows from (4.7) and (4.9) that the sequence $(e^{\omega'}A_m)_{m\in\mathbb{Z}}$ admits an exponential dichotomy with respect to the sequence of norms $\|\cdot\|_{2,m}$. By (4.8) and (4.9), we have

$$\begin{aligned}
e^{\omega'}\|A_nx\|_{2,n+1} &= \|\mathcal{B}'(n+1,n)x\|_{1,n}\\
&\leqslant \|\mathcal{B}'(n+1,n)P_n^2x\|_{2,n+1} + \|\mathcal{B}'(n+1,n)(\mathrm{Id}-P_n^2)x\|_{2,n+1}\\
&\leqslant 2e^{b'+\omega'}\|x\|_{2,n} + e^{-\lambda}\|x\|_{2,n}\\
&\leqslant 3e^{b'+\omega'}\|x\|_{2,n}
\end{aligned}$$

for $x \in X$ and $n \in \mathbb{Z}$. This shows that the second inequality in (4.1) holds with $C_2 = 3e^{b'}$. □

By Lemmas 1 and 6, the sequence $(e^{\omega'}A_m)_{m\in\mathbb{Z}}$ has an admissibility property with respect to the sequence of norms $\|\cdot\|_{2,m}$. Finally, since $\varepsilon < b+d$, one can choose ω and ω' so that $\varepsilon \leqslant \omega - \omega'$. □

Now we establish the converse of Theorem 4.

Theorem 5. *Assume that there exist sequences of norms $\|\cdot\|_{1,m}$ and $\|\cdot\|_{2,m}$ for $m \in \mathbb{Z}$ and constants $D', \omega > 0$, $\varepsilon \geqslant 0$, $C_1, C_2 > 1$ and $\omega' < 0$ with $\varepsilon \leqslant \omega - \omega'$ satisfying properties 1–3 of Theorem 2 and property (4.1) for $n \in \mathbb{Z}$ and $x \in X$. Then the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a strong nonuniform exponential trichotomy.*

Proof. Using the same notation as in the proof of Theorem 3, it follows from (4.1) that

$$\begin{aligned}
\|\mathcal{A}(m,n)P_n^1x\| &\leqslant \|\mathcal{A}(m,n)P_n^1x\|_{1,m}\\
&\leqslant C_1^{n-m}\|P_n^1x\|_{1,n}\\
&\leqslant DC_1^{n-m}\|x\|_{1,n}\\
&\leqslant DD'C_1^{n-m}e^{\varepsilon|n|}\|x\|
\end{aligned}$$

for $m \leqslant n$ and $x \in X$. Similarly,

$$\begin{aligned}
\|\mathcal{A}(m,n)Q_n^2x\| &\leqslant \|\mathcal{A}(m,n)Q_n^2x\|_{2,m}\\
&\leqslant C_2^{m-n}\|Q_n^2x\|_{2,m}\\
&\leqslant DC_2^{m-n}\|x\|_{2,m}\\
&\leqslant DD'C_2^{m-n}e^{\varepsilon|n|}\|x\|
\end{aligned}$$

for $m \geqslant n$ and $x \in X$. In other words, the nonuniform exponential trichotomy given by Theorem 3 is in fact a strong nonuniform exponential trichotomy. □

5. ROBUSTNESS

In this section we establish the persistence of the notions of a nonuniform exponential trichotomy and of a strong nonuniform exponential trichotomy under sufficiently small linear perturbations.

Theorem 6. *Let $(A_m)_{m\in\mathbb{Z}}$ and $(B_m)_{m\in\mathbb{Z}}$ be sequences of invertible linear operators in $B(X)$ such that:*

1. *$(A_m)_{m\in\mathbb{Z}}$ admits a nonuniform exponential trichotomy with $\varepsilon < b+d$;*
2. *there exists $\rho > 0$ such that*

$$\|A_m - B_m\| \leqslant \rho e^{-\varepsilon|m|} \quad \textit{for} \quad m \in \mathbb{Z}. \tag{5.1}$$

If ρ is sufficiently small, then the sequence $(B_m)_{m\in\mathbb{Z}}$ also admits a nonuniform exponential trichotomy.

Proof. We first recall a result established in [1].

Lemma 7. *Let $(A_m)_{m\in\mathbb{Z}}$ and $(B_m)_{m\in\mathbb{Z}}$ be sequences of invertible linear operators in $B(X)$ such that:*

1. *$(A_m)_{m\in\mathbb{Z}}$ admits an exponential dichotomy with respect to a sequence of norms $\|\cdot\|_m$;*
2. *there exists $c > 0$ such that*
$$\|(A_{n-1} - B_{n-1})x\|_n \leqslant c\|x\|_{n-1} \quad \textit{for} \quad n \in \mathbb{Z},\ x \in X.$$

If c is sufficiently small, then the sequence $(B_m)_{m\in\mathbb{Z}}$ admits an exponential dichotomy with respect to the same sequence of norms.

Let $\omega' < 0 < \omega$ be the constants and let $\|\cdot\|_{i,m}$, for $m \in \mathbb{Z}$ and $i = 1, 2$, be the norms given by Theorem 2 (in particular, they satisfy property (3.1)). By (3.1) and (5.1), we have

$$\begin{aligned}\left\|(e^{\omega} A_{n-1} - e^{\omega} B_{n-1})x\right\|_{1,n} &\leqslant D' e^{\omega} e^{\varepsilon|n|} \|(A_{n-1} - B_{n-1})x\| \\ &\leqslant \rho D' e^{\omega-\varepsilon} \|x\|_{1,n-1}\end{aligned}$$

for $x \in X$ and $n \in \mathbb{Z}$. Hence, it follows from Lemmas 1 and 7 that for any sufficiently small ρ, the sequence $(e^{\omega} B_m)_{m\in\mathbb{Z}}$ has an admissibility property with respect to the sequence of norms $\|\cdot\|_{1,m}$. Analogously, for any sufficiently small ρ, the sequence $(e^{\omega'} B_m)_{m\in\mathbb{Z}}$ has an admissibility property with respect to the sequence of norms $\|\cdot\|_{2,m}$. The conclusion of the theorem now follows directly from Theorem 3. □

The following is a version of Theorem 6 for the notion of a strong nonuniform exponential trichotomy.

Theorem 7. *Let $(A_m)_{m\in\mathbb{Z}}$ and $(B_m)_{m\in\mathbb{Z}}$ be sequences of invertible linear operators in $B(X)$ such that:*

1. *$(A_m)_{m\in\mathbb{Z}}$ admits a strong nonuniform exponential trichotomy with $\varepsilon < b + d$;*
2. *there exists $\rho > 0$ such that property (5.1) holds.*

If ρ is sufficiently small, then the sequence $(B_m)_{m\in\mathbb{Z}}$ admits a strong nonuniform exponential trichotomy.

Proof. In view of Theorem 6 and the characterization of a strong nonuniform exponential trichotomy given by Theorems 4 and 5, it is sufficient to show that there exist constants $M_1, M_2 > 1$ such that

$$\frac{1}{M_1}\|x\|_{1,n} \leqslant \|B_n x\|_{1,n+1} \quad \text{and} \quad \|B_n x\|_{2,n+1} \leqslant M_2\|x\|_{2,n} \tag{5.2}$$

for $x \in X$ and $n \in \mathbb{Z}$. It follows from (3.1) and (5.1) that

$$\begin{aligned}\|B_n x\|_{1,n+1} &\geqslant \|A_n x\|_{1,n+1} - \|(A_n - B_n)x\|_{1,n+1} \\ &\geqslant \frac{1}{C_1}\|x\|_{1,n} - \rho D' e^{\varepsilon}\|x\|_{1,n}\end{aligned}$$

and similarly,

$$\begin{aligned}\|B_n x\|_{2,n+1} &\leqslant \|A_n x\|_{2,n+1} + \|(A_n - B_n)x\|_{2,n+1} \\ &\leqslant C_2\|x\|_{2,n} + \rho D' e^{\varepsilon}\|x\|_{2,n}.\end{aligned}$$

Taking ρ sufficiently small, we obtain the inequalities in (5.2) and so it follows from Theorem 5 that the sequence $(B_m)_{m\in\mathbb{Z}}$ admits a strong nonuniform exponential trichotomy. □

6. PARTIALLY HYPERBOLIC SETS

In this section we obtain versions of the results in the former sections for the class of nonuniformly partially hyperbolic sets. This corresponds to considering various trajectories simultaneously instead of a single trajectory. The latter corresponds to considering a sequence of linear operators as in the former sections.

Let M be a compact Riemannian manifold and let $f\colon M \to M$ be a C^1 diffeomorphism. A measurable map $\mathcal{A}$ defined on $M \times \mathbb{Z}$ is said to be a *linear cocycle* over f if for each $x \in M$ and $n, m \in \mathbb{Z}$:

1. $\mathcal{A}(x,n)\colon T_xM \to T_{f^n(x)}M$ is a linear map;
2. $\mathcal{A}(x,0) = \mathrm{Id}$;
3. $\mathcal{A}(x,n+m) = \mathcal{A}(f^m(x),n)\mathcal{A}(x,m)$.

The map $A(x) = \mathcal{A}(x,1)$ is called the *generator* of the cocycle $\mathcal{A}$. We shall always assume that there exists $C > 0$ such that

$$\|A(x)\| \leqslant C \quad \text{and} \quad \|A(x)^{-1}\| \leqslant C$$

for $x \in M$. For example, this holds if the map $x \mapsto A(x)$ is continuous and so if $\mathcal{A}$ is the derivative cocycle (in which case $A(x) = d_xf$).

Now let $\mathcal{A}$ be a linear cocycle over f. An f-invariant measurable set $\Lambda \subset M$ is said to be *nonuniformly partially hyperbolic* with respect to $\mathcal{A}$ if given $\varepsilon > 0$, there exist constants $0 \leqslant a < b$, $0 \leqslant c < d$, splittings

$$T_xM = E^s(x) \oplus E^u(x) \oplus E^c(x)$$

for $x \in \Lambda$, with projections $P^s(x)$, $P^u(x)$ and $P^c(x)$, and measurable functions $C, K\colon \Lambda \to \mathbb{R}^+$ such that for each $x \in \Lambda$, $v \in T_xM$ and $n \in \mathbb{Z}$:

1. $A(x)E^s(x) = E^s(y)$, $A(x)E^u(x) = E^u(y)$ and $A(x)E^c(x) = E^c(y)$, where $y = f(x)$;
2. for $n \geqslant 0$,
$$\|\mathcal{A}(x,n)P^s(x)v\|_{f^n(x)} \leqslant C(x)e^{-dn}e^{\varepsilon n}\|v\|_x \tag{6.1}$$
and
$$\|\mathcal{A}(x,-n)P^u(x)v\|_{f^{-n}(x)} \leqslant C(x)e^{-bn}e^{\varepsilon n}\|v\|_x; \tag{6.2}$$
3.
$$\|\mathcal{A}(x,n)P^c(x)v\|_{f^n(x)} \leqslant C(x)e^{an}e^{\varepsilon n}\|v\|_x$$
and
$$\|\mathcal{A}(x,-n)P^c(x)v\|_{f^{-n}(x)} \leqslant C(x)e^{cn}e^{\varepsilon n}\|v\|_x; \tag{6.3}$$
4.
$$C(f^n(x)) \leqslant C(x)e^{\varepsilon|n|}. \tag{6.4}$$

Moreover, we say that Λ is *nonuniformly hyperbolic* with respect to $\mathcal{A}$ if it is nonuniformly partially hyperbolic and $P^c(x) = 0$ for $n \in \mathbb{Z}$.

Given a norm $\|\cdot\|'$ on the tangent bundle $T_\Lambda M$, for each $x \in \Lambda$ we denote by Y_x the set of all sequences $\boldsymbol{\mu} = (\mu_n)_{n\in\mathbb{Z}}$ with $\mu_n \in T_{x_n}M$ and $x_n = f^n(x)$ for $n \in \mathbb{Z}$, such that

$$\|\boldsymbol{\mu}\| = \sup_{n\in\mathbb{Z}} \|\mu_n\|'_{x_n} < +\infty.$$

One can easily verify that $Y_x = (Y_x, \|\cdot\|)$ is a Banach space. Moreover, we define a linear map R_x by

$$(R_x\boldsymbol{\mu})_n = \mu_n - A(x_{n-1})\mu_{n-1}, \quad n \in \mathbb{Z},$$

on the domain formed by all $\boldsymbol{\mu} = (\mu_n)_{n\in\mathbb{Z}} \in Y_x$ such that $R_x\boldsymbol{\mu} \in Y_x$.

The following two results give a characterization of the notion of a partially hyperbolic set in terms of admissibility properties.

Theorem 8. *Let $\Lambda \subset M$ be a nonuniformly partially hyperbolic set with respect to a cocycle $\mathcal{A}$. Then there exist $\varepsilon_0, \omega, D > 0$ and $\omega' < 0$ and given $\varepsilon \in (0, \varepsilon_0)$, there exist norms $\|\cdot\|' = \|\cdot\|^{\varepsilon,1}$ and $\|\cdot\|'' = \|\cdot\|^{\varepsilon,2}$ on $T_\Lambda M$ and a measurable function $G\colon \Lambda \to \mathbb{R}^+$ such that for each $x \in \Lambda$:*

1. $$\frac{1}{2}\|v\|_x \leqslant \|v\|_x^{\varepsilon,i} \leqslant G(x)\|v\|_x, \quad v \in T_xM,\ i = 1, 2;$$

2. $$G(f^n(x)) \leqslant e^{2\varepsilon|n|}G(x), \quad n \in \mathbb{Z};$$

3. *the map $R_x^1\colon Y_x^1 \to Y_x^1$ defined with respect to the cocycle $\mathcal{B}(x, n) = e^{\omega n}\mathcal{A}(x, n)$ and the norm $\|\cdot\|'$ is well defined, bounded and invertible;*

4. *the map $R_x^2\colon Y_x^2 \to Y_x^2$ defined with respect to the cocycle $\mathcal{B}'(x, n) = e^{\omega' n}\mathcal{A}(x, n)$ and the norm $\|\cdot\|''$ is well defined, bounded and invertible;*

5. $\|(R_x^i)^{-1}\| \leqslant D$ *for $i = 1, 2$.*

Proof. The proof of the theorem is analogous to the proof of Theorem 2. We first recall a result established in [2].

Lemma 8. *Let $\Lambda \subset M$ be a nonuniformly hyperbolic set with respect to a cocycle $\mathcal{A}$. Then there exist $\varepsilon_0, D > 0$ such that given $\varepsilon \in (0, \varepsilon_0)$, there exist a norm $\|\cdot\|' = \|\cdot\|^\varepsilon$ on $T_\Lambda M$ and a measurable function $G\colon \Lambda \to \mathbb{R}^+$ such that for each $x \in \Lambda$:*

1. $$\frac{1}{2}\|v\|_x \leqslant \|v\|_x^{\varepsilon} \leqslant G(x)\|v\|_x, \quad v \in T_xM;$$

2. $G(f^n(x)) \leqslant e^{2\varepsilon|n|}G(x)$ *for $n \in \mathbb{Z}$;*

3. *$R_x\colon Y_x \to Y_x$ is well defined, bounded and invertible;*

4. $\|R_x^{-1}\| \leqslant D$.

Take $\omega \in (c,d)$. It follows from (6.1), (6.2) and (6.3) that for each $x \in \Lambda$, $v \in T_xM$ and $n \geqslant 0$, we have

$$\|\mathcal{B}(x,n)P^s(x)v\|_{f^n(x)} \leqslant C(x)e^{(\omega-d)n}e^{\varepsilon n}\|v\|_x,$$

$$\|B(x,-n)P^u(x)v\|_{f^{-n}(x)} \leqslant C(x)e^{-(b+\omega)n}e^{\varepsilon n}\|v\|_x,$$

and

$$\|B(x,-n)P^c(x)v\|_{f^{-n}(x)} \leqslant C(x)e^{-(\omega-c)n}e^{\varepsilon n}\|v\|_x.$$

This shows that Λ is a nonuniformly hyperbolic set with respect to the cocycle $\mathcal{B}$. Hence, it follows from Lemma 8 that there exists $\varepsilon_0 > 0$ such that given $\varepsilon \in (0, \varepsilon_0)$, there exist a norm $\|\cdot\|' = \|\cdot\|^{\varepsilon,1}$ on $T_\Lambda M$ and a measurable function $G\colon \Lambda \to \mathbb{R}^+$ satisfying properties 1, 2, 3 and 5 in the theorem.

Analogously, take $\omega' \in (-b,-a)$. Then Λ is a nonuniformly hyperbolic set with respect to the cocycle $\mathcal{B}'$. Using again Lemma 8, we obtain norms $\|\cdot\|^{\varepsilon,2}$ satisfying properties 1, 2, 4 and 5 in the theorem. □

Now we establish the converse of Theorem 8.

Theorem 9. *Let $\Lambda \subset M$ be an f-invariant measurable set. Assume that there exist $\varepsilon_0, \omega, D > 0$ and $\omega' < 0$ such that given $\varepsilon \in (0, \varepsilon_0)$, there exist norms $\|\cdot\|^{\varepsilon,1}$ and $\|\cdot\|^{\varepsilon,2}$ on $T_\Lambda M$ and a measurable function $G\colon \Lambda \to \mathbb{R}^+$ satisfying properties* 1–5 *in Theorem* 8*. Then Λ is a nonuniformly partially hyperbolic set with respect to the cocycle $\mathcal{A}$.*

Proof. We need the following result established in [2].

Lemma 9. *Let $\Lambda \subset M$ be an f-invariant measurable set. Assume that there exist $\varepsilon_0, D > 0$ such that given $\varepsilon \in (0, \varepsilon_0)$, there exist a norm $\|\cdot\|^{\varepsilon}$ on $T_\Lambda M$ and a measurable function $G\colon \Lambda \to \mathbb{R}^+$ satisfying properties* 1–4 *of Lemma* 8*. Then Λ is a nonuniformly hyperbolic set with respect to the cocycle $\mathcal{A}$.*

It follows from Lemma 9 that Λ is a nonuniformly hyperbolic set with respect to both cocycles $\mathcal{B}$ and $\mathcal{B}'$. Hence, there exist projections $P^1(x)$ and $P^2(x)$ on T_xM for each $x \in \Lambda$, a constant $\lambda > 0$ and for each $\varepsilon > 0$ a measurable function $C\colon \Lambda \to (0, +\infty)$ satisfying (6.4) such that for each $x \in \Lambda$, $v \in T_xM$ and $n \geqslant 0$:

$$\|\mathcal{B}(x,n)P^1(x)v\|_{f^n(x)} \leqslant C(x)e^{-\lambda n}e^{\varepsilon n}\|v\|_x, \tag{6.5}$$

$$\|\mathcal{B}(x,-n)Q^1(x)v\|_{f^{-n}(x)} \leqslant C(x)e^{-\lambda n}e^{\varepsilon n}\|v\|_x \tag{6.6}$$

and

$$\|\mathcal{B}'(x,n)P^2(x)v\|_{f^n(x)} \leqslant C(x)e^{-\lambda n}e^{\varepsilon n}\|v\|_x, \tag{6.7}$$

$$\|\mathcal{B}'(x,-n)Q^2(x)v\|_{f^{-n}(x)} \leqslant C(x)e^{-\lambda n}e^{\varepsilon n}\|v\|_x, \tag{6.8}$$

where $Q^i(x) = \mathrm{Id} - P^i(x)$.

Lemma 10. *For each $x \in \Lambda$, we have*

$$\operatorname{Im} P^1(x) \subset \operatorname{Im} P^2(x) \quad \text{and} \quad \operatorname{Im} Q^2(x) \subset \operatorname{Im} Q^1(x). \tag{6.9}$$

Proof (of the lemma). Proceeding as in the proof of Proposition 1, we find that

$$E^s(x) = \left\{ v \in T_xM : \sup_{n \geqslant 0} \|\mathcal{A}(x,n)v\|_{f^n(x)} < +\infty \right\} \tag{6.10}$$

and

$$E^u(x) = \left\{ v \in T_xM : \sup_{n \geqslant 0} \|\mathcal{A}(x,-n)v\|_{f^{-n}(x)} < +\infty \right\}. \tag{6.11}$$

Now take $v \in \operatorname{Im} P^1(x)$. We have

$$\|\mathcal{B}'(x,n)v\|_{f^n(x)} = e^{\omega' n}\|\mathcal{A}(x,n)v\| = e^{(\omega'-\omega)n}\|\mathcal{B}(x,n)v\|_{f^n(x)}$$

for $n \geqslant 0$. Hence, the first inclusion in (6.9) follows readily from (6.10). The second inclusion can be obtained in an analogous manner using (6.11). □

Proceeding as in the proof of Theorem 3 (see Lemmas 3 and 4), one can show that $\operatorname{Id} - P^1(x) - Q^2(x)$ is a projection on the tangent space T_xM with range $\operatorname{Im} P^2(x) \cap \operatorname{Im} Q^1(x)$ for each $x \in \Lambda$. It follows from (6.5) and (6.8) that

$$\|\mathcal{A}(x,n)P^1(x)v\|_{f^n(x)} \leqslant C(x)e^{-(\lambda+\omega)n}e^{\varepsilon n}\|v\|_x \tag{6.12}$$

and

$$\|\mathcal{A}(x,-n)Q^2(x)v\|_{f^{-n}(x)} \leqslant C(x)e^{-(\lambda-\omega')n}e^{\varepsilon n}\|v\|_x \tag{6.13}$$

for $x \in \Lambda$, $v \in T_xM$ and $n \geqslant 0$. Similarly, it follows from (6.6) and (6.7) that

$$\|\mathcal{A}(x,-n)v\|_{f^{-n}(x)} \leqslant C(x)e^{-(\lambda-\omega)n}e^{\varepsilon n}\|v\|_x$$

and

$$\|\mathcal{A}(x,n)v\|_{f^n(x)} \leqslant C(x)e^{-(\lambda+\omega')n}e^{\varepsilon n}\|v\|_x$$

for $x \in \Lambda$, $v \in \operatorname{Im} P^2(x) \cap \operatorname{Im} Q^1(x)$ and $n \geqslant 0$. Moreover, by (6.6) and (6.7), we have

$$\|\operatorname{Id} - P^1(x) - Q^2(x)\| \leqslant 3C(x).$$

Hence,

$$\|\mathcal{A}(x,-n)(\operatorname{Id} - P^1(x) - Q^2(x))v\|_{f^{-n}(x)} \leqslant 3C(x)^2 e^{-(\lambda-\omega)n}e^{\varepsilon n}\|v\|_x \tag{6.14}$$

and

$$\|\mathcal{A}(x,n)(\operatorname{Id} - P^1(x) - Q^2(x))v\|_{f^n(x)} \leqslant 3C(x)^2 e^{-(\lambda+\omega')n}e^{\varepsilon n}\|v\|_x \tag{6.15}$$

for $x \in \Lambda$, $v \in T_xM$ and $n \geqslant 0$. It follows from (6.12), (6.13), (6.14) and (6.15) that Λ is a nonuniformly partially hyperbolic set with respect to $\mathcal{A}$. □

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