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Problems in Standard Parallels Reconstruction

Keywords: old maps; map projections; conic projections; standard parallels

Summary: In previous research, we proved that meridians are straight lines and parallels are the arcs of concentric circles, in the map *Nuova carta geografica dello Stato Ecclesiastico* by J. R. Bošković and Ch. Maire, published in 1755. We also assumed the map was produced in a normal aspect conic projection which is equidistant along meridians. However, it is not enough to know the map projection alone. We also need to know its parameters. Therefore, we proposed a new procedure for determining standard parallels of an old map with the normal aspect conic projection which is equidistant along meridians, if the radius of the sphere, which is mapped, is unknown. We calculated that probable standard parallels, in which the map was produced, are at 41°50’ and 44°30’ of latitude. Furthermore, we proposed a procedure for testing reliability of determining standard parallels in the old map. We concluded that determining standard parallels is not an easy task. In this paper, we examine whether our assumption, that the map was produced in the normal aspect conic projection equidistant along meridians, was correct. We calculated rectangular coordinates in an equal-area, conformal, equidistant along parallels and equidistant along meridians normal aspect conic projections for pairs of standard parallels at intervals of 10' which are symmetrical in relation to the central parallel of the mapped area. The final conclusion on the map projection is based on the comparison of calculated average values of deviations between theoretical and depicted graticule on the map.

Even though the Earth's curved surface cannot be mapped into a plane with a linear distortion equal to zero, the effect of linear distortions can be significantly reduced by selecting the most appropriate standard parallels. Linear distortions are more pronounced on small scale maps of a large area. In the last part of the paper, the procedure for determining the bandwidth in which the effect of standard parallels is more pronounced is described.

Introduction

In previous research, we developed a procedure for testing the map projection type in which an old map was produced and a procedure of determining parameters of a normal aspect conic projection which is equidistant along meridians (Triplat Horvat 2014, Triplat Horvat and Lapaine 2014a, Triplat Horvat and Lapaine 2014b, Triplat Horvat and Lapaine 2015). The procedure was tested on the map *Nuova carta geografica dello Stato Ecclesiastico* by J. R. Bošković and Ch. Maire. Furthermore, we concluded that determining projection parameters, particularly standard parallels, in which an old map was produced, is not an easy task, because it does not provide a unique solution, especially for maps that show a small area of the Earth’s surface.

Therefore, we decided to continue our research and examine whether our assumption, that the map was produced in the normal aspect conic projection which is equidistant along meridians, is correct.

Furthermore, considering that the map represents a small area of the Earth's surface, we decided to examine how large the area represented on the map should be so that the effect of

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distortion, caused by selection of the standard parallels, is expressed.

Conic Map Projections

Normal aspect conic projection is a map projection in which meridians are mapped as straight lines intersecting in one point at the angles proportional to the difference between corresponding longitudes of the meridians and parallels are mapped as arcs of concentric circles with the centre in the meridian intersection. General equations for the normal aspect spherical conic projections are (Snyder and Voxland 1989, Frančula 2004):

$$\begin{aligned} x &= \rho_j - \rho\cos\delta \\ y &= \rho\sin\delta \end{aligned} \tag{1}$$

with

$$\begin{aligned} \rho &= \rho(\varphi) \\ \delta &= k(\lambda - \lambda_0) \end{aligned} \tag{2}$$

where

x, y – rectangular coordinates in the projection plane
φ, λ – geographical coordinates of points on a sphere
$\rho = \rho(\varphi)$ – radius of the parallel in the projection plane which corresponds to the latitude φ
δ – angle between meridians in the projection plane
k – proportionality constant ($0 < k < 1$)
λ_0 – longitude of the central meridian of the projection
ρ_j – constant which defines the relationship (a shift) of the polar coordinate system ρ, δ with respect to the rectangular coordinate system *x, y*. It is usually taken as the radius of the parallel in the projection which corresponds to the smallest (southernmost) latitude of the northern hemisphere.

The function $\rho = \rho(\varphi)$ is usually determined according to mapping conditions, such as conformality, equivalency or equidistance.

In this paper, we are going to limit our analysis to projection of the Earth's sphere with radius *R* for numerical examples, because in previous research (Triplat Horvat 2014) we demonstrated that Ch. Maire and J. R. Bošković mapped a sphere onto the projection plane when they produced the map. The method for determining standard parallels for the projection of the Earth's ellipsoid could be a subject of future research.

In all normal aspect conic projections, the linear scale along parallels can be determined using the equation:

$$n = n(\varphi) = \frac{k\rho}{R\cos\varphi}. \tag{3}$$

The linear scale along meridians in all normal aspect conical projections can be determined using the equation:

$$m = m(\varphi) = -\frac{d\rho}{Rd\varphi}. \tag{4}$$

In all normal aspect conic projections, the area scale is defined as (Borčić 1955, Snyder and Voxland 1989, Frančula 2004):

$$p(\varphi) = m(\varphi)n(\varphi)\,. \tag{5}$$

In order for a parallel with latitude φ to be a standard parallel, it has to be that

$$m(\varphi) = n(\varphi) = 1\,. \tag{6}$$

Normal Aspect Conic Projections which are Equidistant along Parallels

From the condition of the equidistance along parallels

$$n(\varphi) = \frac{k\rho}{R\cos\varphi} = 1 \tag{7}$$

it follows that

$$\rho(\varphi) = \frac{R}{k}\cos\varphi\,. \tag{8}$$

In order for a parallel with latitude φ to be a standard parallel, it also has to be that

$$m(\varphi) = -\frac{d\rho}{Rd\varphi} = 1\,. \tag{9}$$

From (8), it follows that

$$\frac{d\rho}{d\varphi} = -\frac{R}{k}\sin\varphi\,. \tag{10}$$

From (9) and (10), it follows that

$$k = \sin\varphi\,. \tag{11}$$

Therefore, if the parallel which corresponds to the latitude φ_1 is a standard parallel in a conic projection which is equidistant along parallels, then in the equation (8)

$$k = \sin\varphi_1\,. \tag{12}$$

Normal Aspect Conic Projections which are Equidistant along Meridians

From the condition of equidistance along meridians

$$m(\varphi) = -\frac{d\rho}{Rd\varphi} = 1 \tag{13}$$

it follows that

$$d\rho = -Rd\varphi\,. \tag{14}$$

After integration

$$\rho(\varphi) = K - R\varphi = R(C - \varphi) \tag{15}$$

where we denoted

$$C = \frac{K}{R}\,. \tag{16}$$

In order for a parallel with latitude ϕ to be a standard parallel, it also has to be that

$$n(\varphi) = \frac{k\rho}{R\cos\varphi} = 1 \tag{17}$$

i.e.

$$k\rho = R\cos\varphi . \tag{18}$$

Considering (15)

$$kC - k\varphi = \cos\varphi . \tag{19}$$

Since we have two constants, *k* and *C*, they can be determined based on two given latitudes φ_1 and φ_2 by solving two equations with two unknowns:

$$\begin{aligned} kC - k\varphi_1 &= \cos\varphi_1 \\ kC - k\varphi_2 &= \cos\varphi_2 . \end{aligned} \tag{20}$$

After subtracting the two equations we get

$$k = \frac{\cos\varphi_1 - \cos\varphi_2}{\varphi_2 - \varphi_1} . \tag{21}$$

However, if we multiply the first equation with φ_2 and the second one with φ_1 and then subtract them we get

$$kC = \frac{\varphi_2\cos\varphi_1 - \varphi_1\cos\varphi_2}{\varphi_2 - \varphi_1} \tag{22}$$

then

$$C = \frac{\varphi_2\cos\varphi_1 - \varphi_1\cos\varphi_2}{\cos\varphi_1 - \cos\varphi_2} , \tag{23}$$

i.e.

$$K = R\frac{\varphi_2\cos\varphi_1 - \varphi_1\cos\varphi_2}{\cos\varphi_1 - \cos\varphi_2} . \tag{24}$$

In the special case of $\varphi_1 = \varphi_2$, *k* is not determined by the relation (21). By a limit of a sequence when $\varphi_2 \to \varphi_1$, it can be obtained

$$k = \sin\varphi_1 \tag{25}$$

$$C = \varphi_1 + \frac{1}{\tan\varphi_1} \tag{26}$$

$$K = R\left(\varphi_1 + \frac{1}{\tan\varphi_1}\right). \tag{27}$$

Normal Aspect Conic Equal-area Projections

From the condition

$$p(\varphi) = m(\varphi)n(\varphi) = 1 \tag{28}$$

it follows

$$\rho d\rho = -\frac{R^2}{k}\cos\varphi d\varphi \ , \tag{29}$$

and by solving the differential equation, we get

$$\rho^2 = K - \frac{2}{k}R^2 \sin\varphi \, , \tag{30}$$

i.e., by introducing constants

$$C = \frac{K}{R^2} \tag{31}$$

$$\rho = R\sqrt{C - \frac{2}{k}\sin\varphi} \ . \tag{32}$$

In order for a parallel with latitude φ to be a standard parallel, due to (28) it is sufficient that

$$n(\varphi) = \frac{k\rho}{R\cos\varphi} = 1 \tag{7}$$

or

$$m(\varphi) = -\frac{d\rho}{Rd\varphi} = 1 \, , \tag{13}$$

i.e.

$$\rho(\varphi) = \frac{R}{k}\cos\varphi \, . \tag{8}$$

Considering (30) and (8), we have a condition for standard parallel corresponding to latitude φ

$$k^2C - 2k\sin\varphi = \cos^2\varphi \, . \tag{36}$$

Since we have two constants k and C we can determine them based on two given latitudes φ_1 and φ_2 by solving two equations with two unknowns:

$$\begin{aligned} k^2C - 2k\sin\varphi_1 &= \cos^2\varphi_1 \\ k^2C - 2k\sin\varphi_2 &= \cos^2\varphi_2 . \end{aligned} \tag{37}$$

If we subtract equations (37), we get

$$k = \frac{1}{2}(\sin\varphi_1 + \sin\varphi_2) \, . \tag{38}$$

If we multiply the first equation from (37) by $\sin\varphi_1$ and the second by $\sin\varphi_2$ and then subtract those equations, we get

$$k^2C = 1 + \sin\varphi_1 \sin\varphi_2 \tag{39}$$

and then

$$C = 4\frac{1+\sin\varphi_1\sin\varphi_2}{(\sin\varphi_1+\sin\varphi_2)^2}, \tag{40}$$

i.e.

$$K = 4R^2\frac{1+\sin\varphi_1\sin\varphi_2}{(\sin\varphi_1+\sin\varphi_2)^2}. \tag{41}$$

In the special case of $\varphi_1=\varphi_2$, it follows from (38) that

$$k=\sin\varphi_1, \tag{42}$$

and from (40), i.e. (41)

$$C=\frac{1+\sin^2\varphi_1}{\sin^2\varphi_1}=1+\frac{1}{k^2} \tag{43}$$

$$K=R^2C=R^2\frac{1+\sin^2\varphi_1}{\sin^2\varphi_1}=R^2\left(1+\frac{1}{k^2}\right). \tag{44}$$

Normal Aspect Conic Conformal Projections

From the conformality condition

$$m(\varphi)=n(\varphi) \tag{45}$$

it follows that

$$\frac{d\rho}{\rho}=-k\frac{d\varphi}{\cos\varphi}=-kdq, \tag{46}$$

where we introduced the isometric latitude q for a sphere as

$$dq=\frac{d\varphi}{\cos\varphi}. \tag{47}$$

Let us recall that by the integration of the differential equation (47) with initial condition $q(0)=0$, we can get the solution written in different ways:

$$\begin{aligned} q(\varphi)&=\ln\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)=-\ln\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)=\frac{1}{2}\ln\frac{1+\sin\varphi}{1-\sin\varphi}=\ln\frac{1+\sin\varphi}{\cos\varphi}= \\ &=\tanh^{-1}(\sin\varphi)=\sinh^{-1}(\tan\varphi)=\cosh^{-1}\left(\frac{1}{\cos\varphi}\right)=\mathrm{gd}^{-1}\varphi=\mathrm{lam}(\varphi), \end{aligned} \tag{48}$$

where gd is Gudermannian, i.e. function which is defined by

$$\varphi=\mathrm{gd}(q). \tag{49}$$

By solving differential equation (46), we obtain

$$\ln\rho=-kq+\ln K=\ln\exp(-kq)+\ln K \tag{50}$$

and from there

$$\rho=K\exp(-kq). \tag{51}$$

or introducing constant

$$C = \frac{K}{R} \tag{52}$$

$$\rho = RC\exp(-kq)\,. \tag{53}$$

In order for a parallel with latitude φ to be a standard parallel, due to (28) it is sufficient that

$$n(\varphi) = \frac{k\rho}{R\cos\varphi} = 1 \tag{7}$$

or

$$m(\varphi) = -\frac{d\rho}{Rd\varphi} = 1\,, \tag{13}$$

i.e.

$$\rho(\varphi) = \frac{R}{k}\cos\varphi = \frac{R}{k\cosh q}\,. \tag{8}$$

Considering (53) and (8), we have a condition for a standard parallel corresponding to latitude φ

$$kK\exp(-kq) = \frac{R}{\cosh q}\,, \tag{54}$$

i.e.

$$kC\cosh q = \exp kq\,. \tag{55}$$

Since we have two constants k and C, we can determine them based on two given latitudes φ_1 and φ_2 by solving two equations with two unknowns:

$$\begin{aligned} kC\cosh q_1 &= \exp kq_1 \\ kC\cosh q_2 &= \exp kq_2. \end{aligned} \tag{56}$$

By dividing equations (56) and using logarithmic, we obtain

$$k = \frac{\ln\cosh q_2 - \ln\cosh q_1}{q_2 - q_1}\,. \tag{57}$$

If we use logarithmic on the equations in (56) and then multiply the first equation by q_2 and the second by q_1 and then subtract those equations, we obtain

$$kC = \exp\frac{q_1\ln\cosh q_2 - q_2\ln\cosh q_1}{q_2 - q_1} \tag{58}$$

and then

$$C = \frac{1}{k}\exp\frac{q_1\ln\cosh q_2 - q_2\ln\cosh q_1}{q_2 - q_1} \tag{59}$$

i.e.

$$K = \frac{R}{k}\exp\frac{q_1\ln\cosh q_2 - q_2\ln\cosh q_1}{q_2 - q_1}\,. \tag{60}$$

Expressions (57), (59) and (60), taking into account (48) can be also written by using latitudes such as:

$$k = \frac{\ln\cos\varphi_1 - \ln\cos\varphi_2}{\tanh^{-1}(\sin\varphi_2) - \tanh^{-1}(\sin\varphi_1)} = \frac{\ln\cos\varphi_1 - \ln\cos\varphi_2}{\ln\tan\left(\frac{\pi}{4} + \frac{\varphi_2}{2}\right) - \ln\tan\left(\frac{\pi}{4} + \frac{\varphi_1}{2}\right)}\,. \tag{61}$$

$$C=\frac{1}{k}\exp\frac{\ln\cos\varphi_1\tanh^{-1}(\sin\varphi_2)-\ln\cos\varphi_2\tanh^{-1}(\sin\varphi_1)}{\tanh^{-1}(\sin\varphi_2)-\tanh^{-1}(\sin\varphi_1)}=$$
$$=\frac{1}{k}\exp\frac{\ln\cos\varphi_1\ln\tan\left(\frac{\pi}{4}+\frac{\varphi_2}{2}\right)-\ln\cos\varphi_2\ln\tan\left(\frac{\pi}{4}+\frac{\varphi_1}{2}\right)}{\ln\tan\left(\frac{\pi}{4}+\frac{\varphi_2}{2}\right)-\ln\tan\left(\frac{\pi}{4}+\frac{\varphi_1}{2}\right)} \qquad (62)$$

i.e.

$$K=\frac{R}{k}\exp\frac{\ln\cos\varphi_1\tanh^{-1}(\sin\varphi_2)-\ln\cos\varphi_2\tanh^{-1}(\sin\varphi_1)}{\tanh^{-1}(\sin\varphi_2)-\tanh^{-1}(\sin\varphi_1)} \qquad (63)$$

In special case of $\varphi_1=\varphi_2$, k is not determined by the relation (21). By a limit of a sequence when $\varphi_2\to\varphi_1$, i.e. when $q_2\to q_1$ it can be obtained that

$$k=\tanh q_1=\sin\varphi_1 \qquad (64)$$

$$C=\frac{\exp(q_1\tanh q_1)}{\sinh q_1}=\frac{\tan^{\sin\varphi_1}\left(\frac{\pi}{4}+\frac{\varphi_1}{2}\right)}{\tan\varphi_1} \qquad (65)$$

$$K=R\frac{\exp(q_1\tanh q_1)}{\sinh q_1}=R\frac{\tan^{\sin\varphi_1}\left(\frac{\pi}{4}+\frac{\varphi_1}{2}\right)}{\tan\varphi_1}. \qquad (66)$$

Determination of the Radius of the Sphere by Using Data Read from the Map

Suppose that the map was produced in a normal aspect conic projection. If we have access to coordinates of n points (y_i,x_i) along the central meridian, in the plane with known geographical coordinates (φ_i,λ_i) of the Earth's sphere (e.g. intersections of the central meridian with drawn parallels), and if we assume we know the latitudes of standard parallels, then it is possible to estimate the size of radius R which comes in the projection equations, and after that, indirectly, the map scale.

Notice that all equations for the function $\rho=\rho(\varphi)$ in a normal aspect conic projection can be written in the form

$$\rho(\varphi)=R\omega(\varphi). \qquad (67)$$

For example, for the conic projection which is equidistant along meridians

$$\omega(\varphi)=C-\varphi, \qquad (68)$$

for the conic equal-area projection

$$\omega(\varphi)=\sqrt{C-\frac{2}{k}\sin\varphi}, \qquad (69)$$

for the conic conformal projection

$$\omega(\varphi)=C\exp(-kq)=C\tan^{k}\left(\frac{\pi}{4}-\frac{\varphi}{2}\right). \qquad (70)$$

Conic Projections with two Standard Parallels

For given values of latitudes φ_1 and φ_2 of standard parallels, parameters *k* and *C* can be calculated for each projection using equations from the previous chapter. Therefore, let us assume that *k* and *C* are known values.

Values $x_i, \varphi_i, i = 1,...,n$ are given. Theoretically, along the central meridian in all normal aspect conic projection, it is true that

$$x_i = \rho_j - \rho_i \,. \tag{71}$$

However, since abscissas x_i are obtained by measuring, the expression (71) generally cannot be fulfilled. Furthermore, since values φ_i can be considered as error free, corrections are introduced so that

$$x_i + v_i = \rho_j - \rho_i \,. \tag{72}$$

Corrections v_i can be determined in different ways, i.e. with different conditions. If we apply the least square method, the task can be formulated as: we are looking for *R* and φ_j so that

$$f = f(R, \varphi_j) = \sum v_i^2 = \sum (\rho_j - \rho_i - x_i)^2 = \min \,. \tag{73}$$

Due to (67), we can write

$$\rho_i = \rho_i(\varphi) = R\omega_i(\varphi) = R\omega_i \tag{74}$$

so (73) becomes

$$f = f(R, \varphi_j) = \sum v_i^2 = \sum (\rho_j - R\omega_i - x_i)^2 = \min \,. \tag{75}$$

Furthermore,

$$\begin{aligned} \frac{\partial f}{\partial \rho_j} &= 2\sum(\rho_j - R\omega_i - x_i) = 0 \\ \frac{\partial f}{\partial R} &= 2\sum(\rho_j - R\omega_i - x_i)(-\omega_i) = 0. \end{aligned} \tag{76}$$

The system (76) of two linear equations with two unknowns can be written as

$$\begin{aligned} n\rho_j - R\sum \omega_i &= \sum x_i \\ \rho_j \sum \omega_i - R\sum \omega_i^2 &= \sum x_i \omega_i . \end{aligned} \tag{77}$$

The solution of (77) is

$$\begin{aligned} R &= \frac{\sum x_i \sum \omega_i - n\sum x_i \omega_i}{n\sum \omega_i^2 - \left(\sum \omega_i\right)^2} \\ \rho_j &= \frac{\sum x_i}{n} + R\frac{\sum \omega_i}{n}. \end{aligned} \tag{78}$$

Conic Projections with One Standard Parallel

In addition to projections which are equidistant along meridians, equal-area and conformal projections, we can also include conic projections which are equidistant along parallels. For that projection, (67) also applies with

$$\omega(\varphi) = \frac{\cos\varphi}{k}. \tag{79}$$

For given values of latitude φ_1 which corresponds to a single standard parallel parameter *k*, i.e. *C* can be calculated for each of the projections by using the equations from the previous chapters. The procedure for determining radius *R* and radii of the "southernmost" parallel ρ_j can be performed by the previously described procedure and by using the final equations (78).

Verification of Assertion about Conic Projection which is Equidistant along Meridians

Nuova carta geografica dello Stato Ecclesiastico by Ch. Maire and J. R. Bošković was published in 1755 and consists of three sheets. The area represented on the map stretches from about 41°12'N to 45°3'N of latitude and represents the area of the former Papal State. The map was produced in the approximate scale of 1:370 000. The map projection type and its parameters had been unknown until recently. In her doctoral dissertation, Triplat Horvat (2014) came to the conclusion that the map was produced in a normal aspect conic projection which is equidistant along meridians, and that standard parallels of the projection were at 41°50'N and 44°30'N of the latitude.

In order to analyse the map, for example to determine its map projection type or assess the accuracy of data depicted on it, it is necessary to know rectangular coordinates of characteristic points on a map. Characteristic points for determining the map projection type and parameters which describe the projection can be rectangular coordinates of a graticule or intersections of meridians and parallels depicted on the map. Due to the nature of the procedure, which is going to be described in this paper, we defined the accuracy estimation of the mathematical base *a priori* as ±1.0–2.0 mm in the map scale.

In order to verify our assumption that the map was produced in the normal aspect conic projection which is equidistant along meridians, we calculated theoretical rectangular coordinates of graticule in an equal-area, conformal, equidistant along parallels and equidistant along meridians normal aspect conic projections for several pairs of standard parallels by using equations from Chapter 2. Our conclusion is based on a comparison of calculated average values of deviations between theoretical and depicted graticule in the map. Pairs of standard parallels were selected at intervals of 10' which are symmetrical in relation to the mean parallel (43°10'N) in the mapped area.

The number of clearly visible intersections of meridians and parallels in the map is *n* = 112. We read the geographical coordinates of each point at the edge of the map. We calculated theoretical coordinates for those 112 points for all map projection types by using each pair of standard parallels and the central meridian $(\lambda_0 = 30°)$. We calculated theoretical rectangular coordinates of the graticule by applying equations in (1) with (2). Function ρ in (2) is determined according to mapping conditions, such as conformality, equivalency or equidistance. Thus, for conformal conic projection the function was calculated using equation (53), for equal-area projection by using equation (32), for conic projection which is equidistant along parallels by using equation

(8) and for projection which is equidistant along meridians by using equation (15). In order to calculate values of function ρ by using those equations for all map projection types, it is necessary to determine radius *R* of the Earth's sphere at the scale of the map. We calculated the radius of sphere R by using equation (78), while $\omega(\varphi)$ was calculated for the conformal projection by using equation (70), for equal-area projection by using equation (69), for projection which is equidistant along meridians by using (68) and for projection which is equidistant along parallels by using (79).
We calculated deviations between the theoretical graticule and the one depicted on the map *Nuova carta geografica dello Stato Ecclesiastico* by using the expression

$$v_i = \sum_{i=1}^{n} \sqrt{\Delta y_i^2 + \Delta x_i^2} \tag{80}$$

for each pair of standard parallels at intervals of 10' in relation to the central parallel depicted on the map. The average values of deviations between theoretical and depicted graticule on the map for each pair of the standard parallels are shown in Table 1 for conic conformal projection, equal-area projection and projection which is equidistant along meridians.
Conical projection which is equidistant along parallels has just one standard parallel in the middle of the represented area in the map and for the *Nuova carta geografica dello Stato Ecclesiastico* it equals 43°10'N of latitude. The average value of the deviations between theoretical and depicted rectangular coordinates of graticule on the map for this map projection type equals 0.0028 m.

Conformal		**Equal-area**		**Equidistant along meridians**	
Standard parallels	**v_i [m]**	**Standard parallels**	**v_i [m]**	**Standard parallels**	**v_i [m]**
$\varphi_1 = 42°20'$ $\varphi_2 = 44°00'$	0.00269	$\varphi_1 = 42°20'$ $\varphi_2 = 44°00'$	0.00269	$\varphi_1 = 42°20'$ $\varphi_2 = 44°00'$	0.0013
$\varphi_1 = 42°10'$ $\varphi_2 = 44°10'$	0.00274	$\varphi_1 = 42°10'$ $\varphi_2 = 44°10'$	0.00273	$\varphi_1 = 42°10'$ $\varphi_2 = 44°10'$	0.0013
$\varphi_1 = 42°00'$ $\varphi_2 = 44°20'$	0.00299	$\varphi_1 = 42°00'$ $\varphi_2 = 44°20'$	0.00298	$\varphi_1 = 42°00'$ $\varphi_2 = 44°20'$	0.0013
$\varphi_1 = 41°50'$ $\varphi_2 = 44°30'$	0.00348	$\varphi_1 = 41°50'$ $\varphi_2 = 44°30'$	0.00347	$\varphi_1 = 41°50'$ $\varphi_2 = 44°30'$	0.0013
$\varphi_1 = 41°40'$ $\varphi_2 = 44°40'$	0.00422	$\varphi_1 = 41°40'$ $\varphi_2 = 44°40'$	0.00420	$\varphi_1 = 41°40'$ $\varphi_2 = 44°40'$	0.0013
$\varphi_1 = 41°30'$ $\varphi_2 = 44°50'$	0.00525	$\varphi_1 = 41°30'$ $\varphi_2 = 44°50'$	0.00523	$\varphi_1 = 41°30'$ $\varphi_2 = 44°50'$	0.0013
$\varphi_1 = 41°20'$ $\varphi_2 = 45°00'$	0.00664	$\varphi_1 = 41°20'$ $\varphi_2 = 45°00'$	0.00661	$\varphi_1 = 41°20'$ $\varphi_2 = 45°00'$	0.0013

Table 1: Average values of the distances of theoretical and depicted graticule on the map *Nuova carta geografica dello Stato Ecclesiastico* for pairs of standard parallels at intervals of 10' in relation to the middle parallel of the mapped area.

From the average values of deviations shown in Table 1, we can conclude that *Nuova carta geografica dello Stato Ecclesiastico* was really produced in the normal aspect conic projection which is equidistant along meridians. Average values of deviations between the theoretical graticule and the one depicted on the map are the least in that projection type and are equal to 1.3 mm in the map scale for each pair of standard parallels. Average values of deviations between the theoretical graticule and the one depicted on the map for conic conformal and equal-area projection differ from standard parallels to standard parallels whose values reaches from about 3 to about 7 mm in the map scale. Such deviations, even the smallest ones, exceed the *a priori* defined accuracy estimation.

Determining the Size of an Area in which the Effect of Standard Parallels is More Pronounced

The effect of distortion on various selections of standard parallels can be easily seen in Fig. 1. Figure 1c shows the overlap of graticules in the normal aspect equidistant conic projection with two standard parallels when standard parallels are at 20° and 70° of the latitude and 40° and 50° of the latitude. The area represented in the figure stretches from 0° to 90° of latitude and from –90° to 90° of longitude. It is easy to notice that because the projection is equidistant along meridians, meridians are mapped without distortions (as shown by condition (13)), while the choice of standard parallel affects the angle at which meridians intersect. Equidistant conical projection is a good choice for areas on an Earth's sphere at middle latitudes. Fig. 1 shows that the impact of standard parallels, which are symmetrical, relative to the mean latitude of the mapped area on the shape of the graticule, and linear scale, is relatively small, especially when the area shown on the map does not cover a large range by the longitude. The fact indicates that determination of standard parallels, based on the measured points, and the length of the graticule on the maps will be relatively unreliable.

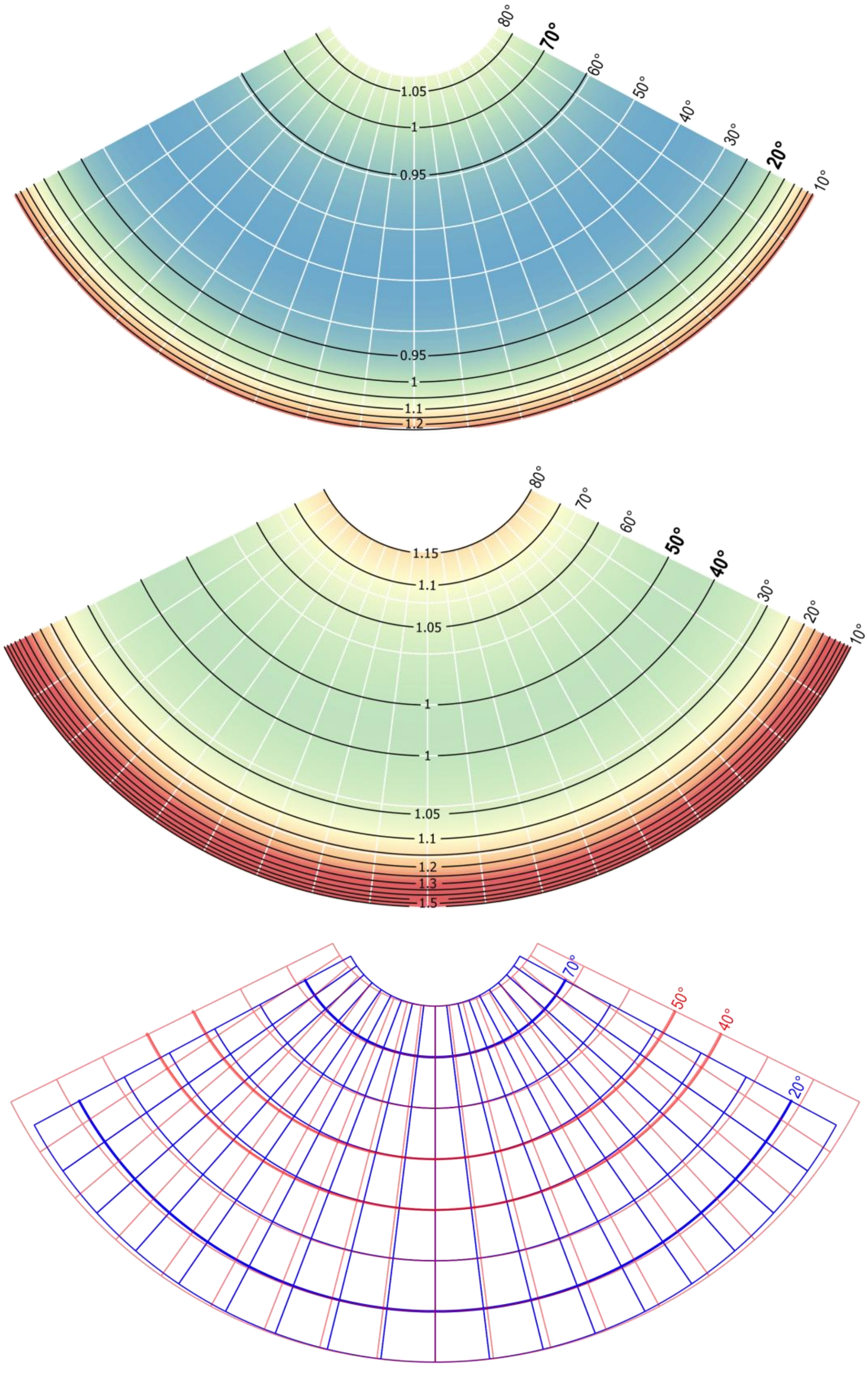

Figure 1. Comparison of linear scales along parallels and shape of graticule in equidistant conic projection for the area of middle latitudes (45°) with two symmetrical and different choices of standard parallels.

We cannot conclude which parallels are definitive standard parallels from the calculated average values of deviations in Chapter 4 (see Table 1). One of our assumptions for the obtained results is that the area represented on the map is small, so the effect of distortion caused by selection

of standard parallels is not more pronounced. Therefore, we decided to examine how large the area represented on the map has to be, so that distortion caused by inappropriate selection of standard parallels is more pronounced.
In order to find a solution for the given task, it is necessary to find the length of the parallel arc for the given standard parallel so that extreme linear distortion in the direction of parallel is as small as possible or that they do not exceed a predefined accuracy.
In order to use the proposed procedure to determine the optimal area in which distortion is not going to be pronounced, an equation is necessary for the linear scale in the normal aspect conic projection which is equidistant along meridians. We are only going to describe the case when two standard parallels φ_1 and φ_2 are given, i.e. parallels along which the linear scale is equal to 1.
Linear scale along parallels in the normal aspect conic projection, which is equidistant along meridians, with two standard parallels φ_1 and φ_2 is given by:

$$n(\varphi)=\frac{k\rho}{R\cos\varphi}, \tag{3}$$

where

$$\rho(\varphi)=R(C-\varphi), \tag{15}$$

$$k=\frac{\cos\varphi_1-\cos\varphi_2}{\varphi_2-\varphi_1} \tag{21}$$

$$C=\frac{\varphi_2\cos\varphi_1-\varphi_1\cos\varphi_2}{\cos\varphi_1-\cos\varphi_2}. \tag{23}$$

Linear distortion is going to be calculated using linear scale according to the expression

$$d=n-1. \tag{81}$$

Let us denote
φ_j ... latitude of the southernmost parallel of the mapped area
φ_s ... latitude of the northernmost parallel of the mapped area
In Chapter 2.2, we showed that projection parameters can be calculated based on latitudes of two standard parallels φ_1 and φ_2. Due to the definition of a standard parallel, it would be natural to take that φ_1 and φ_2 are somewhere within the mapped area, i.e. that

$$\varphi_j\le\varphi_1\le\varphi_2\le\varphi_s \tag{82}$$

although there are some different choices.
A choice of φ_1 and φ_2 according to (82) can be done in an infinite number of ways (e.g. Snyder 1978). For example, for an arbitrary area stretching from 20°N to 70°N of latitude, standard parallels are chosen symmetrically to the middle parallel of the research area (45°N) at intervals of 5°. The results are shown in Fig. 2.

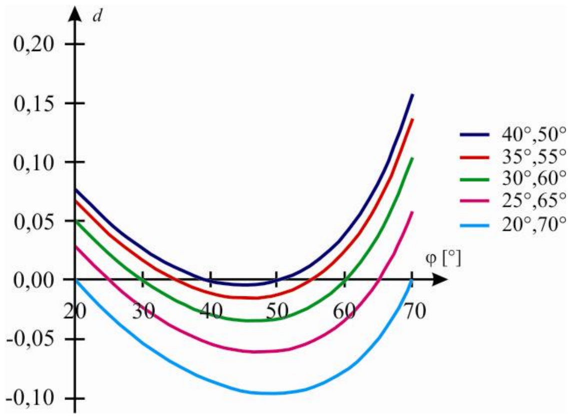

Figure 2. Linear distortion in the normal aspect equidistant projection for an area from 20°N to 70°N of latitude for different standard parallels.

In case when standard parallels are given at 40° and 50° of latitude, the curve of linear distortion is moved almost entirely up, i.e. only a small portion of the distortion takes a negative value (Figure 2). For each following pair of standard parallels, the curve moves down until the curve for standard parallels at 20° and 70° of latitude does not move entirely down and all values of linear distortion have a negative sign. Figure 2 also shows certain asymmetry of the distortion curve because the distortion is not equal on the northern and southern part of the observed area, except if standard parallels are at the edge of the area.

It seems more natural to require that the distribution of distortion in relation to the northern and southern part of an area be symmetrical, if possible and as much as possible. Therefore, we can set the first condition that the linear scale at the edge of an area (along northern and southern parallel) is equal:

$$n(\varphi_j)=n(\varphi_s) \tag{83}$$

by using expression (83), it leads to

$$\frac{k\rho(\varphi_j)}{R\cos\varphi_j}=\frac{k\rho(\varphi_s)}{R\cos\varphi_s}$$

from which first

$$\frac{\rho(\varphi_j)}{\cos\varphi_j}=\frac{\rho(\varphi_s)}{\cos\varphi_s}$$

then, taking into account (15)

$$\frac{C-\varphi_j}{\cos\varphi_j}=\frac{C-\varphi_s}{\cos\varphi_s}$$

and

$$C=\frac{\varphi_s\cos\varphi_j-\varphi_j\cos\varphi_s}{\cos\varphi_j-\cos\varphi_s}. \tag{84}$$

The result of the described procedure are “parabolas” shown in Fig. 3, 4 and 5 for the area studied in the previous example, the area represented in *Charta* by Rigas Velestinlis and the area represented in the Maire's and Bošković's map *Nuova carta…,* where the second parameter k is determined indirectly by choosing different values for φ_1, and by using equation

$$k = \frac{\cos\varphi_1}{C - \varphi_1}\,. \tag{85}$$

In Fig. 3, 4 and 5, we can notice different distribution of the linear scale (n) for each analysed area. Thus, in research area, which stretches from the parallel at 20°N to 70°N of latitude, the linear scale varies from 0.904 to 1.106, for *Charta* it varies from 0.996 to 1.004, while for *Nuova carta* ...it varies from 0.999 to 1.001.

Accordingly, different distributions of linear distortion along parallels can be obtained: from the case in which all distortions have positive signs (except for $\varphi_1=\varphi_2$, when along the parallel the distortion equals 0) or all parallels in the projection are longer than those on the sphere to the case in which all distortions in the mapped area have negative signs (except for $\varphi_1=\varphi_j$ and $\varphi_1=\varphi_s$, when along the parallels the distortion equals 0), or all parallels in the projection are shorter than on the sphere. The common feature of all distributions is that linear distortion along the southernmost and northernmost parallel are equal.

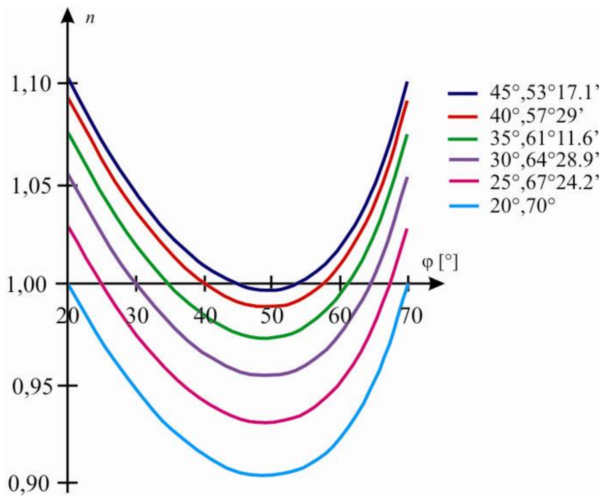

Figure 3. Linear scales in the normal aspect equidistant projection for the area from 20° to 70° of latitude with different standard parallels.

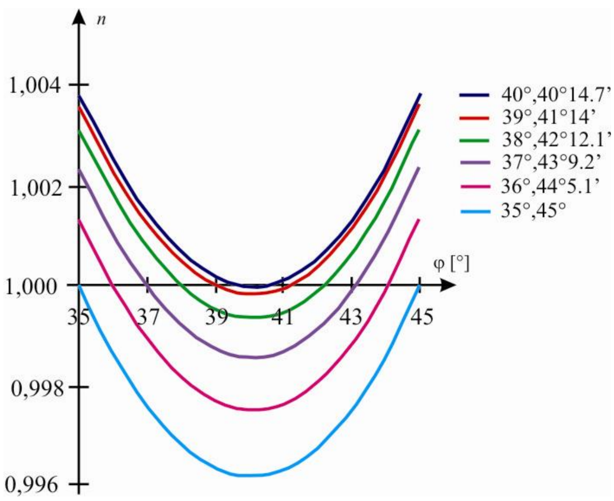

Figure 4. Linear scales in the normal aspect equidistant projection for *Charta* by Rigas Velestinlis with different standard parallels.

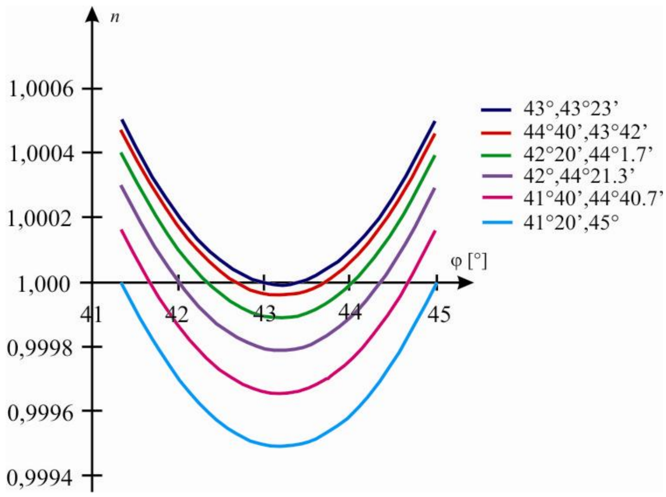

Figure 5. Linear scales in the normal aspect equidistant projection for Nuova carta... by Maire and Bošković with different standard parallels.

In any case, distortion is the largest on marginal parallels, while it is the smallest in the middle part. It seems natural to require a new condition according to which linear distortions on marginal parallels would in absolute values be equal to linear distortions in the middle area, or where they are the least. Let us denote the latitude of parallel along which a linear scale is the least with φ_0, i.e.

$$n(\varphi_0) = \min \qquad (86)$$

then the condition is

$$n(\varphi_s) - 1 = 1 - n(\varphi_0) \text{ or } \frac{n(\varphi_s) + n(\varphi_0)}{2} = 1 \qquad (87)$$

In order for latitude φ_0 to satisfy the condition (86), it has to be that

$$\frac{dn}{d\varphi} = 0 \qquad (88)$$

with (3) and (15), it means that it has to be that

$$C - \varphi_0 = \cot \varphi_0 \qquad (89)$$

and

$$n(\varphi_0) = \frac{k}{\sin \varphi_0}. \qquad (90)$$

Equation (89) is a nonlinear equation which can be solved by one of the known methods for solving equations with C determined by (84). By using the bisection method, we obtained that for the stretching from 20° to 70° of latitude, φ_0 equals 49.2588°, for Charta it equals 40.1226° and for Maire and Bošković's map it equals 43.185°.

Since

$$n(\varphi_s) = \frac{k(C - \varphi_s)}{\cos \varphi_s} \qquad (91)$$

from (90) and (91) can be derived

$$n(\varphi_s) = n(\varphi_0)\sin\varphi_0 \frac{C-\varphi_s}{\cos\varphi_s} \tag{92}$$

and taking into account (87)

$$n(\varphi_0) = \frac{2}{1+\dfrac{\sin\varphi_0}{\sin\dfrac{\varphi_s+\varphi_j}{2}} \dfrac{\dfrac{\varphi_s-\varphi_j}{2}}{\sin\dfrac{\varphi_s-\varphi_j}{2}}} \tag{93}$$

$$n(\varphi_s) = n(\varphi_j) = \frac{2}{1+\dfrac{\sin\dfrac{\varphi_s+\varphi_j}{2}}{\sin\varphi_0} \dfrac{\sin\dfrac{\varphi_s-\varphi_j}{2}}{\dfrac{\varphi_s-\varphi_j}{2}}}\,. \tag{94}$$

It can easily be verified that (87) is true, i.e.

$$n(\varphi_s) + n(\varphi_0) = 2\,. \tag{95}$$

By applying equations (93) and (94), we calculated the values of linear scale for the southernmost and the northernmost latitude and values of linear scale for latitude φ_0 for area from 20° to 70° of latitude, *Charta* by Rigas Velestinlis and for *Nuova carta…* by Maire and Bošković. Results are shown in Table 2, 3 and 4.

Let us denote

s ... length of parallel arc on the sphere

S ... projection length of the parallel arc in normal aspect projection which is equidistant along meridians.

The difference in length of the parallel arc in projection and on the sphere in absolute value is

$$|S-s| = s\left|\frac{S}{s}-1\right| = s\,|n(\varphi)-1|\,, \tag{96}$$

and in a map scale, if the scale is 1:*M*

$$\frac{|S-s|}{M} = \frac{s}{M}|n(\varphi)-1| = \frac{R\cos\varphi\Delta\lambda}{M}|n(\varphi)-1|\,. \tag{97}$$

We are going to obtain the maximum value of (97) under the assumption that $\varphi=\varphi_j$.

The parameter $\Delta\lambda$ in the expression (97) is the longitude difference between the easternmost and the westernmost point of the studied area. For an arbitrary area, we considered the area between 0° and 50° of longitude of the Greenwich prime meridian. *Charta* represents the area between 32°45' and 47°45' of longitude in relation to the Ferro prime meridian or from 15° to 30° in relation to the Greenwich prime meridian. *Nuova carta…* represents the area between 28°40' and 31°20' of longitude in relation to the Ferro prime meridian.

Example 1

The area from 20° to 70° of latitude

R = 6 370 km

M = 8 000 000

φ_j	φ_s	φ_0	$n(\varphi_j)=n(\varphi_s)$	$n(\varphi_0)$	$\Delta\lambda$	$\max\frac{\lvert S-s\rvert}{M}$ [mm]
20°	70°	49.258°	1.050451	0.949549	50°	32.9

Table 2. Values of linear scale and maximum values of differences of parallel arc length in the projection and on the sphere for the area from 20° to 70° of latitude.

Example 2.
Rigas Velestinlis *Charta*
R = 6 370 km
M = 600 000

φ_j	φ_s	φ_0	$n(\varphi_j)=n(\varphi_s)$	$n(\varphi_0)$	$\Delta\lambda$	$\max\frac{\lvert S-s\rvert}{M}$ [mm]
35°	45°	40.123°	1.001915	0.998085	15°	4.4

Table 3. Values of linear scale and maximum values of differences of parallel arc length in the projection and on the sphere for the area represented on *Charta* by Rigas Velestinlis.

Example 3.
Maire and Bošković *Nuova carta…*
R = 6 370 km
M = 370 000

φ_j	φ_s	φ_0	$n(\varphi_j)=n(\varphi_s)$	$n(\varphi_0)$	$\Delta\lambda$	$\max\frac{\lvert S-s\rvert}{M}$ [mm]
41°20'	45°	43.185°	1.000256	0.999744	2°40'	0.2

Table 4. Values of linear scale and maximum values of differences of parallel arc length in the projection and on the sphere for the area on *Nuova carta…* by Maire and Bošković.

We only considered the "ideal" case of distribution of linear distortions in the normal aspect conic projection which equidistant along the meridians. We can also examine borderline cases which could potentially be considered.

We denoted parallel arc length on the sphere with s. Then $\frac{s}{M}$ is arc length of the parallel on a map which is produced in the normal aspect conic projection which is equidistant along meridians in the scale 1:M without taking into account projection distortion. Arc length of a parallel in the projection, taking into account projection distortion, is going to be equal to $\frac{s}{M}n$. The question is whether we can observe/measure the difference on a map

$$\left|\frac{s}{M}n-\frac{s}{M}\right|=\frac{s}{M}\left|n-1\right|.$$

The answer is that we can, if

$$\frac{s}{M}\left|n-1\right|>\varepsilon, \tag{98}$$

where ε is given, for example ε = 0.1 mm or ε = 1 mm, depending on the assumption *a priori.* In her doctoral dissertation, Triplat Horvat (2014) defined the accuracy estimation *a priori* equal to

±1.0–2.0 mm. In the present paper, we decided it is ε = 1 mm. For *s* we took the arc length of the southernmost latitude shown on the map, and we calculated it by using equation $R\cos\varphi_j\Delta\lambda$. We calculated parameter *n* by using equation (3) and taking into account the one which differs the most from 1 on the given area. It is one of $n(\varphi_s), n(\varphi_j), n(\varphi_0)$, depending which pair of standard parallels was chosen for standard parallels. Results are shown in Table 5.
It can be seen from values in Table 5 that differences in the parallel arc length in the projection plane and on the sphere do not exceed the accuracy estimation *a priori* only for the map produced by Maire and Bošković. This confirms our assumption that the area represented on the map is small enough that the effect caused by map projection would not be pronounced. Therefore, we cannot determine standard parallels of the map with certainty.

	Standard parallels	$\max(n(\varphi)-1)$	$\frac{s}{M}\lvert n-1\rvert$ [mm]
Area from 20° to 70° of latitude	$\varphi_1=\varphi_2=\varphi_0=49°15.5'$	0.106271	69.4
	$\varphi_1=20°0'$ $\varphi_2=70°0'$	-0.096062	62.7
Charta	$\varphi_1=\varphi_2=\varphi_0=40°7.4'$	0.003822	8.7
	$\varphi_1=35°0'$ $\varphi_2=45°0'$	-0.003808	8.7
Nuova carta…	$\varphi_1=\varphi_2=\varphi_0=43°11.1'$	0.000512	0.3
	$\varphi_1=41°20'$ $\varphi_2=45°0'$	-0.000512	0.3

Table 5. Differences in the parallel arc length in the projection plane and on the sphere with the influencing deformities when *n* deviates from 1 and the given standard parallel the most.

Charta by Rigas Velestinlis represents the area of 10° of longitude. From values in Table 5 we can see that differences in the parallel arc length are from 5.4 to 8.7 mm in the map scale. Therefore, it attention should be paid to the effect of distortion.
Between the two borderline cases for each of three areas shown in Table 5, there are all others, shown in Figure 3, 4 and 5, as well as the „ideal" cases of linear distortion distribution from examples 1, 2 and 3.

Conclusion

In the paper, we examined the assumption that the map *Nuova carta geografica dello Stato Ecclesiastico* has been produced in the normal aspect conic projection equidistant along meridians. To test it, in the first part of the paper, we recommended the procedure which analyses the type of conic map projection according to the distortion types (equidistant, conform, equivalent) in which a certain old map had been made. In order to execute the procedure, it is needed to know the sphere radius as well. Through the survey that had been carried out, we confirmed the premise that the map had been made in conic projection, equidistant along the meridians.
Considering that the curved Earth's surface cannot be mapped on the projection plane without the deformations, in the second part of the paper we decided to investigate how large the area in the map has to be displayed, so the impact of the deformations cannot be measured on the map, if the map is produced in conic projection equidistant along the meridians.
We tested the procedure in three area magnitudes. The first area stretches between 20° and 70° latitude and 0° to 50° longitude for map zone of the *Charta* Rigasa Velestinlis and the zone *Nuova carta…* by Ch. Maire and J. R. Bošković. The procedure consists of comparison of the parallel arc length in projection plane and on the sphere. This research showed that the procedure

of determining the size of an area does not give a single solution. In order to note/measure the difference between the parallel arc length on the map and on the projection plane, we need to know the map's scale.

We concentrated the paper around the analysis of Earth sphere projection, while the generalization of the Earth's rotational ellipsoid could be an issue in the future research.

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