COMBINATORIAL BASES OF BASIC MODULES FOR AFFINE LIE ALGEBRAS $C_n^{(1)}$

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Abstract. J. Lepowsky and R. L. Wilson initiated the approach to combinatorial Rogers-Ramanujan type identities via vertex operator constructions of standard (i.e. integrable highest weight) representations of affine Kac-Moody Lie algebras. A. Meurman and M. Primc developed further this approach for $\mathfrak{sl}(2,\mathbb{C})\tilde{}$ by using vertex operator algebras and Verma modules. In this paper we use the same method to construct combinatorial bases of basic modules for affine Lie algebras of type $C_n^{(1)}$ and, as a consequence, we obtain a series of Rogers-Ramanujan type identities. A major new insight is a combinatorial parametrization of leading terms of defining relations for level one standard modules for affine Lie algebra of type $C_n^{(1)}$.

1. Introduction

J. Lepowsky and R. L. Wilson [LW] initiated the approach to combinatorial Rogers-Ramanujan type identities via vertex operator constructions of representations of affine Kac-Moody Lie algebras. In [MP1] this approach is developed further for $\mathfrak{sl}(2,\mathbb{C})\tilde{}$ by using vertex operator algebras and Verma modules. In this paper we use the same method to construct combinatorial bases for basic modules of affine Lie algebra of type $C_n^{(1)}$.

The starting point in [MP1] is a PBW spanning set of a standard (i.e., integrable highest weight) module $L(\Lambda)$ of level k, which is then reduced to a basis by using the relation

$$x_\theta(z)^{k+1} = 0 \quad \text{on} \quad L(\Lambda).$$

In [MP1] this relation was interpreted in terms of vertex operator algebras and it was proved for any level k standard module of any untwisted affine Kac-Moody Lie algebra.

After a PBW spanning set is reduced to a basis, it remains to prove its linear independence. The main ingredient of the proof is a combinatorial use of relation

$$x_\theta(z)\tfrac{d}{dz}(x_\theta(z)^{k+1}) = (k+1)x_\theta(z)^{k+1}\tfrac{d}{dz}x_\theta(z)$$

for the annihilating field $x_\theta(z)^{k+1}$. This relation was also interpreted in terms of vertex operator algebras.

By following ideas developed in [MP1] and [MP2], in [P1] and [P2] a general construction of relations for annihilating fields is given by using vertex operator algebras, and by using these relations the problem of constructing combinatorial bases of standard modules is split into a "combinatorial part of the problem" and a "representation theory part of the problem ".

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In this paper we use these results to construct combinatorial bases of basic modules for affine Lie algebras of type $C_n^{(1)}$. A major new insight is a combinatorial parametrization in [PŠ] of leading terms of defining relations for all standard modules for affine Lie algebra of type $C_n^{(1)}$. This is, hopefully, an important step towards a solution of "combinatorial part of the problem" of constructing combinatorial bases of standard modules for affine Lie algebras.

In first nine sections we give a detailed outline of ideas and results involved in this approach, we introduce notation and recall necessary general results from [P1] and [P2]. The results from [P1] on relations among relations are formulated in "untwisted setting"—this may alleviate using the results which are quite technical in "twisted setting". In Section 10 we prove Proposition 10.1 which is the starting point of our construction of combinatorial basis of the basic module $L(\Lambda_0)$ for affine Lie algebra of type $C_n^{(1)}$. In Section 11 we prove linear independence of combinatorial bases by using the combinatorial result from [PŠ] for counting the number of two-embeddings. As a consequence, in Section 12 we obtain a series of combinatorial Rogers-Ramanujan type identities.

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2. Vertex algebras and generating fields

Two formal Laurent series $a(z) = \sum a_n z^{-n-1}$ and $b(z) = \sum b_n z^{-n-1}$, with coefficients in some associative algebra, are said to be mutually local if for some non-negative integer N

$$(z_1 - z_2)^N a(z_1) b(z_2) = (z_1 - z_2)^N b(z_2) a(z_1).$$

A vertex algebra V is a vector space equipped with a specified vector $\mathbf{1}$ called the vacuum vector, a linear operator D on V called the derivation and a linear map

$$V \to (\operatorname{End} V)[[z^{-1}, z]], \qquad v \mapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}$$

satisfying the following conditions for $u, v \in V$:

$$u_n v = 0 \qquad \text{for } n \text{ sufficiently large,} \tag{2.1}$$

$$[D, Y(u, z)] = Y(Du, z) = \frac{d}{dz} Y(u, z), \tag{2.2}$$

$$Y(\mathbf{1}, z) = \mathrm{id}_V \qquad \text{(the identity operator on } V\text{),} \tag{2.3}$$

$$Y(u, z)\mathbf{1} \in (\operatorname{End} V)[[z]] \quad \text{and} \quad \lim_{z \to 0} Y(u, z)\mathbf{1} = u, \tag{2.4}$$

$$Y(u, z) \text{ and } Y(v, z) \text{ are mutually local.} \tag{2.5}$$

Haisheng Li showed [L] that this definition of vertex algebra is equivalent to the original one given by R. E. Borcherds [B]. The formal Laurent series $Y(u, z)$ is called the vertex operator (field) associated with the vector (state) u, and (2.4) gives a state-field correspondence. For coefficients of vertex operators $Y(u, z)$ and $Y(v, z)$ we have the commutator formula

$$[u_m, v_n] = \sum_{i \geq 0} \binom{m}{i} (u_i v)_{m+n-i}. \tag{2.6}$$

Let M be a vector space and $a(z)$ and $b(z)$ two formal Laurent series with coefficients in $\operatorname{End} M$ such that for each $w \in M$

$$a_m w = 0 \quad \text{and} \quad b_m w = 0 \qquad \text{for } m \text{ sufficiently large.} \tag{2.7}$$

Then for each integer n we have a well defined product

$$a(z)_n b(z) = \operatorname{Res}_{z_1}\left((z_1 - z)^n a(z_1) b(z) - (-z + z_1)^n b(z) a(z_1)\right), \tag{2.8}$$

with the convention that $(z_1 - z)^n = z_1^n (1 - z/z_1)^n$ denotes a series obtained by the binomial formula for $(1-\zeta)^n$. If we think of a vertex algebra as a vector space given $\mathbf{1}$, D and multiplications $u_n v$, satisfying (2.1)–(2.5), then we can state the theorem on generating fields due to Haisheng Li [L]:

Theorem 2.1. *A family of mutually local formal Laurent series with coefficients in $\operatorname{End} M$, satisfying (2.7), generates a vertex algebra with the vacuum $\mathbf{1} = id_M$, the derivation $D = \frac{d}{dz}$ and the multiplications $a(z)_n b(z)$.*

A vertex operator algebra (see [FLM]) is a vertex algebra V with a conformal vector ω such that $Y(\omega, z) = \sum L_n z^{-n-2}$ gives the Virasoro algebra operators L_n, with $L_{-1} = D$. It is also required that L_0 defines a $\mathbb{Z}$-grading $V = \coprod V_n$ truncated from below with finite-dimensional eigenspaces V_n.

For $u \in V_n$ we write $\operatorname{wt} u = n$. We shall sometimes use another convention for writing coefficients of vertex operators,

$$Y(u, z) = \sum_{n \in \mathbb{Z}} u(n) z^{-n-\operatorname{wt} u},$$

so that $u(n)$ is a homogeneous operator on the graded space V of degree n.

For a vertex operator algebra V we have a vertex operator algebra structure on $V \otimes V$ with fields $Y(u \otimes v, z) = Y(u,z) \otimes Y(v,z)$ and the conformal vector $\omega \otimes \mathbf{1} + \mathbf{1} \otimes \omega$ (see [FHL]).

3. Vertex algebras for affine Lie algebras

Let $\mathfrak{g}$ be a simple complex Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and $\langle \ , \ \rangle$ a symmetric invariant bilinear form on $\mathfrak{g}$. Via this form we identify $\mathfrak{h}$ with $\mathfrak{h}^*$ and we assume that $\langle \theta, \theta \rangle = 2$ for the maximal root θ (with respect to some fixed basis of the root system). Set

$$\hat{\mathfrak{g}} = \coprod_{j \in \mathbb{Z}} \mathfrak{g} \otimes t^j + \mathbb{C}c, \qquad \tilde{\mathfrak{g}} = \hat{\mathfrak{g}} + \mathbb{C}d.$$

Then $\tilde{\mathfrak{g}}$ is the associated untwisted affine Kac-Moody Lie algebra (cf. [K]) with the commutator

$$[x(i), y(j)] = [x,y](i+j) + i\delta_{i+j,0}\langle x, y\rangle c.$$

Here, as usual, $x(i) = x \otimes t^i$ for $x \in \mathfrak{g}$ and $i \in \mathbb{Z}$, c is the canonical central element, and $[d, x(i)] = ix(i)$. Sometimes we shall denote $\mathfrak{g} \otimes t^j$ by $\mathfrak{g}(j)$. We identify $\mathfrak{g}$ and $\mathfrak{g}(0)$. Set

$$\tilde{\mathfrak{g}}_{<0} = \coprod_{j<0} \mathfrak{g} \otimes t^j, \qquad \tilde{\mathfrak{g}}_{\le 0} = \coprod_{j \le 0} \mathfrak{g} \otimes t^j + \mathbb{C}d, \qquad \tilde{\mathfrak{g}}_{\ge 0} = \coprod_{j \ge 0} \mathfrak{g} \otimes t^j + \mathbb{C}d.$$

For $k\in\mathbb{C}$ denote by $\mathbb{C}v_k$ the one-dimensional $(\tilde{\mathfrak{g}}_{\geq 0}+\mathbb{C}c)$-module on which $\tilde{\mathfrak{g}}_{\geq 0}$ acts trivially and c as the multiplication by k. The affine Lie algebra $\tilde{\mathfrak{g}}$ gives rise to the vertex operator algebra (see [FZ] and [L], here we use the notation from [MP1])

$$N(k\Lambda_0)=U(\tilde{\mathfrak{g}})\otimes_{U(\tilde{\mathfrak{g}}_{\geq 0}+\mathbb{C}c)}\mathbb{C}v_k$$

for level $k\neq -g^\vee$, where $g^\vee$ is the dual Coxeter number of $\mathfrak{g}$; it is generated by the fields

$$x(z)=\sum_{n\in\mathbb{Z}}x_nz^{-n-1},\qquad x\in\mathfrak{g}, \tag{3.1}$$

where we set $x_n=x(n)$ for $x\in\mathfrak{g}$. By the state-field correspondence we have

$$x(z)=Y(x(-1)\mathbf{1},z)\quad\text{for } x\in\mathfrak{g}.$$

The $\mathbb{Z}$-grading is given by $L_0=-d$.

From now on we shall fix the level $k\in\mathbb{Z}_{>0}$, and we shall often denote by V the vertex operator algebra structure on the generalized Verma $\tilde{\mathfrak{g}}$-module $N(k\Lambda_0)$.

4. A completion of the enveloping algebra

Let $\mathcal{U}=U(\hat{\mathfrak{g}})/(c-k)$, where $U(\hat{\mathfrak{g}})$ is the universal enveloping algebra of $\hat{\mathfrak{g}}$ and $(c-k)$ is the ideal generated by the element $c-k$. Note that $\tilde{\mathfrak{g}}$-modules of level k are $\mathcal{U}$-modules. Note that $U(\hat{\mathfrak{g}})$ is graded by the derivation d, and so is the quotient $\mathcal{U}$. Let us denote the homogeneous components of the graded algebra $\mathcal{U}$ by $\mathcal{U}(n)$, $n\in\mathbb{Z}$. We take

$$W_p(n)=\sum_{i\geq p}\mathcal{U}(n-i)\mathcal{U}(i),\qquad p\in\mathbb{Z}_{>0}\ , \tag{4.1}$$

to be a fundamental system of neighborhoods of $0\in\mathcal{U}(n)$. It is easy to see that we have a Hausdorff topological group $(\mathcal{U}(n),+)$, and we denote by $\overline{\mathcal{U}(n)}$ the corresponding completion, introduced in [FZ] (cf. also [H], [FF], and [KL]). Then

$$\overline{\mathcal{U}}=\coprod_{n\in\mathbb{Z}}\overline{\mathcal{U}(n)}$$

is a topological ring.

The definition (4.1) of a fundamental system of neighborhoods is so designed that the product $a(z)_nb(z)$ of two formal Laurent series with coefficients in $\overline{\mathcal{U}}$ is well defined by the formula (2.8). Haisheng Li's arguments in the proof of Theorem 2.1 apply literally and we have:

Proposition 4.1. *The family of mutually local formal Laurent series (3.1) with coefficients in $\overline{\mathcal{U}}$ generates a vertex algebra V' with the vacuum $1\in\overline{\mathcal{U}}$, the derivation $D=\frac{d}{dz}$ and the multiplications $a(z)_nb(z)$. Moreover, the linear map*

$$Y\colon x(-1)\mathbf{1}\mapsto x(z)\qquad \textit{for } x\in\mathfrak{g}$$

extends uniquely to an isomorphism $Y\colon V\to V'$ of vertex operator algebras.

The map

$$Y\colon V\to\overline{\mathcal{U}}\,[[z,z^{-1}]],\qquad v\mapsto Y(v,z)=\sum_{n\in\mathbb{Z}}v_nz^{-n-1},$$

was first constructed by I. B. Frenkel and Y. Zhu in [FZ, Definition 2.2.2] by using another method. From now on we shall consider the coefficients v_n of $Y(v,z)$ for

$v \in V$ as elements in the completion $\overline{\mathcal{U}}$. Then for any highest weight $\tilde{\mathfrak{g}}$-module M of level k the elements $v_n \in \overline{\mathcal{U}}$ act on M, defining a representation of the vertex operator algebra V on M.

By following the notation in [FF] we set

$$U_{\mathrm{loc}} = \mathbb{C}\text{-span}\{v_n \mid v \in V, n \in \mathbb{Z}\} \subset \overline{\mathcal{U}} .$$

From the commutator formula (2.6) we see that U_{loc} is a Lie subalgebra. Let us denote by U the associative subalgebra of $\overline{\mathcal{U}}$ generated by U_{loc}. By construction we have $\mathcal{U} \subset U$. Clearly

$$U = \coprod_{n\in\mathbb{Z}} U(n),$$

where $U(n) \subset U$ is the homogeneous subspace of degree n.

5. Annihilating fields of standard modules

For the fixed positive integer level k the generalized Verma $\tilde{\mathfrak{g}}$-module $N(k\Lambda_0)$ is reducible, and we denote by $N^1(k\Lambda_0)$ its maximal $\tilde{\mathfrak{g}}$-submodule. By [K, Corollary 10.4] the submodule $N^1(k\Lambda_0)$ is generated by the singular vector $x_\theta(-1)^{k+1}\mathbf{1}$, where x_θ is a root vector in $\mathfrak{g}$. Set

$$R = U(\mathfrak{g})x_\theta(-1)^{k+1}\mathbf{1}, \qquad \bar{R} = \mathbb{C}\text{-span}\{r_n \mid r \in R, n \in \mathbb{Z}\}.$$

Then $R \subset N^1(k\Lambda_0)$ is an irreducible $\mathfrak{g}$-module, and $\bar{R} \subset U$ is the corresponding loop $\tilde{\mathfrak{g}}$-module for the adjoint action given by the commutator formula (2.6).

We have the following theorem (see [DL], [FZ], [L], [MP1]):

Theorem 5.1. *Let M be a highest weight $\tilde{\mathfrak{g}}$-module of level k. The following are equivalent:*

(1) *M is a standard module,*
(2) *$\bar{R}$ annihilates M.*

This theorem implies that for a dominant integral weight Λ of level $\Lambda(c) = k$ we have

$$\bar{R}M(\Lambda) = M^1(\Lambda),$$

where $M^1(\Lambda)$ denotes the maximal submodule of the Verma $\tilde{\mathfrak{g}}$-module $M(\Lambda)$. Furthermore, since R generates the vertex algebra ideal $N^1(k\Lambda_0) \subset V$, vertex operators $Y(v, z)$, $v \in N^1(k\Lambda_0)$, annihilate all standard $\tilde{\mathfrak{g}}$-modules $L(\Lambda) = M(\Lambda)/M^1(\Lambda)$ of level k.

We shall call the elements $r_n \in \bar{R}$ *relations* (for standard modules), and $Y(v, z)$, $v \in N^1(k\Lambda_0)$, *annihilating fields* (of standard modules). It is clear that the field

$$Y(x_\theta(-1)^{k+1}\mathbf{1}, z) = x_\theta(z)^{k+1}$$

generates all annihilating fields.

6. Tensor products and induced representations

The vertex operator algebra V has a Lie algebra structure with the commutator

$$[u, v] = u_{-1}v - v_{-1}u = \sum_{n\geq 0}(-1)^n D^{(n+1)}(u_n v), \tag{6.1}$$

and $\tilde{\mathfrak{g}}_{<0}\mathbf{1}$ is a Lie subalgebra. Moreover, the map

$$\tilde{\mathfrak{g}}_{<0}\mathbf{1} \to \tilde{\mathfrak{g}}_{<0}, \qquad u \mapsto u_{-1},$$

is a Lie algebra isomorphism and we have the "adjoint" action

$$u_{-1}\colon v \mapsto [u,v]$$

of the Lie algebra $\tilde{\mathfrak{g}}_{<0}$ on V. Since L_{-1}, L_0 and y_0, $y \in \mathfrak{g}$, are derivations of the product $u_{-1}v$, they are also derivations of the bracket $[u,v]$, and we can extend the "adjoint" action of the Lie algebra $\tilde{\mathfrak{g}}_{<0}$ on V to the "adjoint" action of the Lie algebra

$$\mathbf{C}L_{-1} \ltimes \tilde{\mathfrak{g}}_{\leq 0} \cong \big(\mathbf{C}L_{-1} + \mathbf{C}L_0 + \mathfrak{g}(0)\big) \ltimes \tilde{\mathfrak{g}}_{<0}\mathbf{1}.$$

The subspace

$$\bar{R}\,\mathbf{1} = \coprod_{i=0}^{\infty} D^i R \subset V$$

is a $\tilde{\mathfrak{g}}_{\geq 0}$-submodule invariant for the action of $D = L_{-1}$. Then the right hand side of (6.1) implies that $\bar{R}\,\mathbf{1}$ is invariant for the "adjoint" action of $\mathbf{C}L_{-1} \ltimes \tilde{\mathfrak{g}}_{\leq 0}$, we shall denote it by $(\bar{R}\,\mathbf{1})_{\mathrm{ad}}$.

Hence we have the induced $\tilde{\mathfrak{g}}$-module $U(\tilde{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{g}}_{\geq 0}+\mathbb{C}c)} \bar{R}\,\mathbf{1}$ and the tensor product $(\bar{R}\,\mathbf{1})_{\mathrm{ad}} \otimes V$ of $(\mathbf{C}L_{-1} \ltimes \tilde{\mathfrak{g}}_{\leq 0})$-modules, and we have two maps

$$\Psi\colon U(\tilde{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{g}}_{\geq 0}+\mathbb{C}c)} \bar{R}\,\mathbf{1} \to N(k\Lambda_0), \qquad u \otimes w \mapsto uw,$$

$$\Phi\colon (\bar{R}\,\mathbf{1})_{\mathrm{ad}} \otimes V \to V, \qquad u \otimes w \mapsto u_{-1}w.$$

Note that the map Ψ is a homomorphism of $\tilde{\mathfrak{g}}$-modules, and that Ψ intertwines the actions of L_{-1} and L_0. Hence, by restriction, Ψ is a $(\mathbf{C}L_{-1} \ltimes \tilde{\mathfrak{g}}_{\leq 0})$-module map. The following theorem relates $\ker\Phi$ with induced representations of $\tilde{\mathfrak{g}}$:

Theorem 6.1. *(i) There is a unique isomorphism of* $(\mathbf{C}L_{-1} \ltimes \tilde{\mathfrak{g}}_{\leq 0})$*-modules*

$$\Xi\colon (\bar{R}\,\mathbf{1})_{ad} \otimes V \to U(\tilde{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{g}}_{\geq 0}+\mathbb{C}c)} \bar{R}\,\mathbf{1}$$

such that $\Xi(w \otimes \mathbf{1}) = 1 \otimes w$ *for all* $w \in \bar{R}\,\mathbf{1}$.

(ii) The map Φ *is a homomorphism of* $(\mathbf{C}L_{-1} \ltimes \tilde{\mathfrak{g}}_{\leq 0})$*-modules and* $\Phi = \Psi \circ \Xi$. *In particular,* $\ker\Phi$ *is a* $(\mathbf{C}L_{-1} \ltimes \tilde{\mathfrak{g}}_{\leq 0})$*-module and*

$$\Xi(\ker\Phi) = \ker\Psi.$$

We call elements in $\ker\Phi$ *relations for annihilating fields* (cf. [P1], [P2]) since

$$\sum \colon Y(a,z)Y(b,z)\colon = 0 \qquad \text{for} \qquad \sum a \otimes b \in \ker\Phi.$$

By Theorem 6.1 we may identify the relations for annihilating fields with elements of $\ker\Psi$, which is easier to study by using the representation theory of affine Lie algebras.

7. Generators of relations for annihilating fields

Let $\{x^i\}_{i\in I}$ and $\{y^i\}_{i\in I}$ be dual bases in $\mathfrak{g}$. For $r \in R$ we define Sugawara's relation

$$q_r = \frac{1}{k+g^\vee}\sum_{i\in I} x^i(-1) \otimes y^i(0)r - 1 \otimes Dr \tag{7.1}$$

as an element of $U(\tilde{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{g}}_{\geq 0}+\mathbb{C}c)} \bar{R}\,\mathbf{1}$. As in the case of Casimir operator, Sugawara's relation q_r does not depend on a choice of dual bases $\{x^i\}_{i\in I}$ and $\{y^i\}_{i\in I}$.

Proposition 7.1. *(i)* q_r *is an element of* $\ker\Psi$.
(ii) $r \mapsto q_r$ *is a* $\mathfrak{g}$*-module homomorphism from* R *into* $\ker\Psi$.
(iii) $x(i)q_r = 0$ *for all* $x \in \mathfrak{g}$ *and* $i > 0$.

Let us denote the set of all Sugawara's relations (7.1) by

$$Q_{\text{Sugawara}} = \{q_r \mid r \in R\} \subset \ker\Psi,$$

and let us define the $\tilde{\mathfrak{g}}$-module homomorphism

$$\Psi_0 \colon U(\tilde{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{g}}_{\geq 0}+\mathbb{C}c)} R \to N(k\Lambda_0), \qquad u \otimes w \mapsto uw.$$

Then we have:

Proposition 7.2. *As a* $(\mathbf{C}L_{-1} \ltimes \tilde{\mathfrak{g}}_{\leq 0})$*-module* $\ker\Psi$ *is generated by*

$$\ker\Psi_0 + Q_{Sugawara}\,.$$

Let us denote by α_* all simple roots of $\tilde{\mathfrak{g}}$ connected with α_0 in a Dynkin diagram:

$$\alpha_* \neq \alpha_0\,, \qquad \langle \alpha_0, \alpha_*^\vee\rangle \neq 0.$$

For $A_n^{(1)}$, $n \geq 2$, there are exactly two such simple roots, for all the other untwisted affine Lie algebras $\tilde{\mathfrak{g}}$ there is exactly one such simple root. In the case $\tilde{\mathfrak{g}} \ncong \mathfrak{sl}(2,\mathbb{C})\tilde{}$ we have a root vector $x_{\theta-\alpha_*} = [x_{-\alpha_*}, x_\theta]$ in the corresponding finite-dimensional $\mathfrak{g}$.

Since R generates the maximal $\tilde{\mathfrak{g}}$-submodule $N^1(k\Lambda_0)$ of $N(k\Lambda_0)$, we have the exact sequence of $\tilde{\mathfrak{g}}$-modules

$$U(\tilde{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{g}}_{\geq 0}+\mathbb{C}c)} R \xrightarrow{\Psi_0} N(k\Lambda_0) \to L(k\Lambda_0) \to 0.$$

Generators of $\ker\Psi_0$ can be determined by using Garland-Lepowsky's resolution

$$\cdots \to E_2 \to E_1 \to E_0 \to L(k\Lambda_0) \to 0$$

of a standard module in terms of generalized Verma modules [GL], or by using the BGG type resolution of a standard module in terms of Verma modules, due to A. Rocha-Caridi and N. R. Wallach [RW]:

Proposition 7.3. *Let* $\tilde{\mathfrak{g}} \ncong \mathfrak{sl}(2,\mathbb{C})\tilde{}$ *be an untwisted affine Lie algebra. Then* $\ker\Psi_0$ *is generated by the singular vector(s)*

$$x_{\theta-\alpha_*}(-1) \otimes x_\theta(-1)^{k+1}\mathbf{1} - x_\theta(-1) \otimes x_{\theta-\alpha_*}(-1)x_\theta(-1)^k\mathbf{1}, \quad \langle \alpha_0, \alpha_*^\vee\rangle \neq 0.$$

By combining Theorem 6.1 and Propositions 7.2 and 7.3 we have a description of generators of relations for annihilating fields:

Theorem 7.4. *Let* $\tilde{\mathfrak{g}} \ncong \mathfrak{sl}(2,\mathbb{C})\tilde{}$ *be an untwisted affine Lie algebra. Then the* $(\mathbf{C}L_{-1} \ltimes \tilde{\mathfrak{g}}_{\leq 0})$*-module* $\ker\Phi$ *is generated by vectors*

$$x_\theta(-1)^{k+1}\mathbf{1} \otimes x_{\theta-\alpha_*}(-1)\mathbf{1} - x_{\theta-\alpha_*}(-1)x_\theta(-1)^k\mathbf{1} \otimes x_\theta(-1)\mathbf{1}, \quad \langle \alpha_0, \alpha_*^\vee\rangle \neq 0,$$
$$\tfrac{1}{k+g^\vee}\sum_{i\in I} y^i(0)x_\theta(-1)^{k+1}\mathbf{1} \otimes x^i(-1)\mathbf{1} + L_{-1}\big(\tfrac{1}{k+g^\vee}\,\Omega - 1\big)x_\theta(-1)^{k+1}\mathbf{1} \otimes \mathbf{1}.$$

This description of generators of relations for annihilating fields has some disadvantages when it comes to combinatorial applications. Namely, the obvious relation

$$x_\theta(z)^{k+1}\tfrac{d}{dz}x_\theta(z) - \tfrac{1}{k+1}\,\tfrac{d}{dz}(x_\theta(z)^{k+1})x_\theta(z) = 0$$

for the annihilating field $x_\theta(z)^{k+1}$ comes from the element

$$q_{(k+2)\theta} = x_\theta(-2) \otimes x_\theta(-1)^{k+1}\mathbf{1} - x_\theta(-1) \otimes x_\theta(-2)x_\theta(-1)^k\mathbf{1} \tag{7.2}$$

in $\ker\Psi$. This element $q_{(k+2)\theta}$ has length $k+2$ in the natural filtration, but when written in terms of generators described in Theorem 7.4, it is expressed in terms of elements of length $>k+2$. On the other hand, we can obtain from (7.2) both the singular vector(s)

$$q_{(k+2)\theta-\alpha_*} = x_{\theta-\alpha_*}(-1)\otimes x_\theta(-1)^{k+1}\mathbf{1} - x_\theta(-1)\otimes x_{\theta-\alpha_*}(-1)x_\theta(-1)^k\mathbf{1}$$

in $\ker\Psi_0$ and the Sugawara singular vector

$$q_{(k+1)\theta} = \tfrac{1}{k+g^\vee}\sum_{i\in I} x^i(-1)\otimes y^i(0)x_\theta(-1)^{k+1}\mathbf{1} - 1\otimes Dx_\theta(-1)^{k+1}\mathbf{1}$$

by using the action of $\tilde{\mathfrak{g}}_{\geq 0}$ on $\ker\Psi$:

Lemma 7.5. *Let Ω be the Casimir operator for $\mathfrak{g}\not\cong\mathfrak{sl}(2,\mathbb{C})$ and $\lambda=(k+2)\theta-\alpha_*$. Then*

$$\begin{aligned} q_{(k+2)\theta-\alpha_*} &= x_{-\alpha_*}(1)q_{(k+2)\theta},\\ q_{(k+1)\theta} &= \tfrac{k+1}{2(k+2)(k+g^\vee)}\big(\Omega-(\lambda+2\rho,\lambda)\big)x_{-\theta}(1)q_{(k+2)\theta}. \end{aligned}$$

For any untwisted affine Lie algebra $\tilde{\mathfrak{g}}$, including $\mathfrak{sl}(2,\mathbb{C})\tilde{}$, the $(\mathbf{C}L_{-1}\ltimes\tilde{\mathfrak{g}})$-module $\ker\Psi$ is generated by the vector $q_{(k+2)\theta}$. This generator plays an important role in combinatorial applications.

8. Leading terms

The associative algebra $\mathcal{U}=U(\hat{\mathfrak{g}})/(c-k)$ inherits from $U(\hat{\mathfrak{g}})$ the filtration $\mathcal{U}_\ell$, $\ell\in\mathbb{Z}_{\geq 0}$; let us denote by $\mathcal{S}\cong S(\bar{\mathfrak{g}})$ the corresponding commutative graded algebra.

Let B be a basis of $\mathfrak{g}$. We fix the basis $\tilde{B}$ of $\tilde{\mathfrak{g}}$,

$$\tilde{B}=\bar{B}\cup\{c,d\},\quad \bar{B}=\bigcup_{j\in\mathbb{Z}} B\otimes t^j,$$

so that $\bar{B}$ may also be viewed as a basis of $\bar{\mathfrak{g}}=\hat{\mathfrak{g}}/\mathbb{C}c$. Let $\preceq$ be a linear order on $\bar{B}$ such that

$$i<j \quad\text{implies}\quad x(i)\prec y(j).$$

The symmetric algebra $\mathcal{S}$ has a basis $\mathcal{P}$ consisting of monomials in basis elements $\bar{B}$. Elements $\pi\in\mathcal{P}$ are finite products of the form

$$\pi=\prod_{i=1}^{\ell} b_i(j_i),\quad b_i(j_i)\in\bar{B},$$

and we shall say that π is a colored partition of degree $|\pi|=\sum_{i=1}^{\ell} j_i\in\mathbb{Z}$ and length $\ell(\pi)=\ell$, with parts $b_i(j_i)$ of degree j_i and color b_i. We shall usually assume that parts of π are indexed so that

$$b_1(j_1)\preceq b_2(j_2)\preceq\cdots\preceq b_\ell(j_\ell).$$

We associate with a colored partition π its shape $\operatorname{sh}\pi$, the "plain" partition

$$j_1\leq j_2\leq\cdots\leq j_\ell.$$

The basis element $1\in\mathcal{P}$ we call the colored partition of degree 0 and length 0, we may also denote it by $\varnothing$, suggesting it has no parts. The set of all colored partitions of degree n and length ℓ is denoted as $\mathcal{P}^\ell(n)$. The set of all colored partitions with parts $b_i(j_i)$ of degree $j_i<0$ (respectively $j_i\leq 0$) is denoted as $\mathcal{P}_{<0}$ (respectively $\mathcal{P}_{\leq 0}$).

Note that $\mathcal{P} \subset \mathcal{S}$ is a monoid with the unit element 1, the product of monomials κ and ρ is denoted by $\kappa\rho$. For colored partitions κ, ρ and $\pi = \kappa\rho$ we shall write $\kappa = \pi/\rho$ and $\rho \subset \pi$. We shall say that $\rho \subset \pi$ is an embedding (of ρ in π), notation suggesting that π "contains" all the parts of ρ.

We shall fix a monomial basis

$$u(\pi) = b_1(j_1)b_2(j_2)\dots b_n(j_\ell), \quad \pi \in \mathcal{P},$$

of the enveloping algebra $\mathcal{U}$.

Clearly $\bar{B} \subset \mathcal{P}$, viewed as colored partitions of length 1. We assume that on $\mathcal{P}$ we have a linear order $\preceq$ which extends the order $\preceq$ on $\bar{B}$. Moreover, we assume that order $\preceq$ on $\mathcal{P}$ has the following properties:

- $\ell(\pi) > \ell(\kappa)$ implies $\pi \prec \kappa$.
- $\ell(\pi) = \ell(\kappa)$, $|\pi| < |\kappa|$ implies $\pi \prec \kappa$.
- Let $\ell(\pi) = \ell(\kappa)$, $|\pi| = |\kappa|$. Let π be a partition $b_1(j_1) \preceq b_2(j_2) \preceq \cdots \preceq b_\ell(j_\ell)$ and κ a partition $a_1(i_1) \preceq a_2(i_2) \preceq \cdots \preceq a_\ell(i_\ell)$. Then $\pi \preceq \kappa$ implies $j_\ell \leq i_\ell$.
- Let $\ell \geq 0$, $n \in \mathbb{Z}$ and let $S \subset \mathcal{P}$ be a nonempty subset such that all π in S have length $\ell(\pi) \leq \ell$ and degree $|\pi| = n$. Then S has a minimal element.
- $\mu \preceq \nu$ implies $\pi\mu \preceq \pi\nu$.
- The relation $\pi \prec \kappa$ is a well order on $\mathcal{P}_{\leq 0}$.

Remark 8.1. An order with these properties is used in [MP1]; colored partitions are compared first by length and degree, and then by comparing degrees of parts and colors of parts in the reverse lexicographical order.

For $\pi \in \mathcal{P}$, $|\pi| = n$, set

$$U^{\mathcal{P}}_{[\pi]} = \overline{\mathbb{C}\text{-span}\{u(\pi') \mid |\pi'| = |\pi|, \pi' \succeq \pi\}}\ ,$$

$$U^{\mathcal{P}}_{(\pi)} = \overline{\mathbb{C}\text{-span}\{u(\pi') \mid |\pi'| = |\pi|, \pi' \succ \pi\}}\ ,$$

the closure taken in $\overline{\mathcal{U}(n)}$. Set

$$U^{\mathcal{P}}(n) = \bigcup_{\pi\in\mathcal{P},\ |\pi|=n} U^{\mathcal{P}}_{[\pi]}, \qquad U^{\mathcal{P}} = \coprod_{n\in\mathbb{Z}} U^{\mathcal{P}}(n) \subset \overline{\mathcal{U}}.$$

The construction of $U^{\mathcal{P}}$ depends on a choice of $(\mathcal{P}, \preceq)$. Since by assumption $\mu \preceq \nu$ implies $\pi\mu \preceq \pi\nu$, we have that $U^{\mathcal{P}}$ is a subalgebra of $\overline{\mathcal{U}}$. Moreover, we have a sequence of subalgebras:

Proposition 8.2. $\mathcal{U} \subset U \subset U^{\mathcal{P}} \subset \overline{\mathcal{U}}$.

As in [MP1], we have:

Lemma 8.3. *For $\pi \in \mathcal{P}$ we have $U^{\mathcal{P}}_{[\pi]} = \mathbb{C}u(\pi) + U^{\mathcal{P}}_{(\pi)}$. Moreover,*

$$\dim U^{\mathcal{P}}_{[\pi]}/U^{\mathcal{P}}_{(\pi)} = 1.$$

For $u \in U^{\mathcal{P}}_{[\pi]}$, $u \notin U^{\mathcal{P}}_{(\pi)}$ we define the *leading term*

$$\ell t(u) = \pi.$$

Proposition 8.4. *Every element $u \in U^{\mathcal{P}}(n)$, $u \neq 0$, has a unique leading term $\ell t(u)$.*

By Proposition 8.4 every nonzero homogeneous u has the unique leading term. For a nonzero element $u \in U^{\mathcal{P}}$ we define the leading term $\ell t(u)$ as the leading term of the nonzero homogeneous component of u of smallest degree. For a subset $S \subset U^{\mathcal{P}}$ set

$$\ell t(S) = \{\ell t(u) \mid u \in S, u \neq 0\}.$$

We are interested mainly in leading terms of elements in $U \subset U^{\mathcal{P}}$, which have the following properties:

Proposition 8.5. *For all $u, v \in U^{\mathcal{P}} \backslash \{0\}$ we have $\ell t(uv) = \ell t(u) \ell t(v)$.*

Proposition 8.6. *Let $W \subset U^{\mathcal{P}}$ be a finite-dimensional subspace and let $\ell t(W) \to W$ be a map such that*

$$\rho \mapsto w(\rho), \qquad \ell t\big(w(\rho)\big) = \rho.$$

Then $\{w(\rho) \mid \rho \in \ell t(W)\}$ is a basis of W.

Since R is finite-dimensional, the space $\bar{R} \subset U$ is a direct sum of finite-dimensional homogeneous subspaces. Hence Proposition 8.6 implies that we can parametrize a basis of $\bar{R}$ by the set of leading terms $\ell t(\bar{R})$: we fix a map

$$\ell t(\bar{R}) \to \bar{R}, \quad \rho \mapsto r(\rho) \quad \text{such that} \quad r(\rho) \in U(|\rho|), \ \ell t(r(\rho)) = \rho,$$

then $\{r(\rho) \mid \rho \in \ell t(\bar{R})\}$ is a basis of $\bar{R}$. We will assume that this map is such that the coefficient C of "the leading term" $u(\rho)$ in "the expansion" of $r(\rho) = Cu(\rho) + \dots$ is chosen to be $C = 1$. Note that our assumption $R \subset N^1(k\Lambda_0)$ implies that $1 \notin \ell t(\bar{R})$ and that $\ell t(\bar{R}) \cdot \mathcal{P}$ is a proper ideal in the monoid $\mathcal{P}$.

For an embedding $\rho \subset \pi$, where $\rho \in \ell t(\bar{R})$, we define the element $u(\rho \subset \pi)$ in U by

$$u(\rho \subset \pi) = u(\pi/\rho) r(\rho).$$

9. A rank theorem

Let $a \in V$ be a homogeneous element. Then we have

$$Y(a, z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n - \mathrm{wt}\, a}, \qquad a(n) \in U(n).$$

If M is a level k highest weight $\tilde{\mathfrak{g}}$-module, then the action of coefficients $a(n)$ on M makes M a V-module with vertex operators

$$Y_M(a, z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n - \mathrm{wt}\, a}, \qquad a(n) \in \mathrm{End}\, M.$$

Then $M \otimes M$ is a $V \otimes V$-module. For a homogeneous element $q = a \otimes b$ the vertex operator is defined by

$$Y_{M \otimes M}(q, z) = Y_M(a, z) \otimes Y_M(b, z) = \sum_{n \in \mathbb{Z}} \Big(\sum_{i+j=n} a(i) \otimes b(j) \Big) z^{-n - \mathrm{wt}\, a - \mathrm{wt}\, a}.$$

Since the condition (2.7) is satisfied, the coefficient

$$q(n) = \sum_{i+j=n} a(i) \otimes b(j)$$

is a well defined operator on $M\otimes M$. On the other hand, we want to make sense of this formula for $a(i), b(j)\in U$, where the condition (2.7) is replaced by the convergence in the completion $\overline{\mathcal{U}}$. For this reason set

$$\big(U\bar\otimes U\big)(n)=\prod_{i+j=n}\big(U(i)\otimes U(j)\big),\qquad U\bar\otimes U=\coprod_{n\in\mathbb{Z}}\big(U\bar\otimes U\big)(n).$$

The elements of $U\bar\otimes U$ are finite sums of homogeneous sequences in $U\otimes U$, we shall denote them as $\sum_{i+j=n}a(i)\otimes b(j)$. For a fixed $n\in\mathbb{Z}$ we have a linear map

$$\chi(n)\colon V\otimes V\to\big(U\bar\otimes U\big)(n)$$

defined for homogeneous elements a and b by

$$\chi(n)\colon a\otimes b\mapsto\sum_{p+r=n}a(p)\otimes b(r).$$

We think of $\chi(n)(q)$ as "the coefficient $q(n)$ of the vertex operator $Y(q,z)$". We shall write $q(n)=\chi(n)(q)$ for an element $q\in V\otimes V$ and $Q(n)=\chi(n)(Q)$ for a subspace $Q\subset V\otimes V$.

Since we have the adjoint action of $\hat{\mathfrak g}$ on U, we define "the adjoint action" of $\hat{\mathfrak g}$ on $U\bar\otimes U$ by

$$[x(m),\sum_{p+r=n}a(p)\otimes b(r)]=\sum_{p+r=n}[x(m),a(p)]\otimes b(r)+\sum_{p+r=n}a(p)\otimes[x(m),b(r)].$$

Note that we have the action of $\hat{\mathfrak g}$ on $V\otimes V$ given by

$$x_i(a\otimes b)=(x_ia)\otimes b+a\otimes(x_ib)\,,\qquad x\in\mathfrak g,\ i\in\mathbb{Z}.$$

As expected, we have the following commutator formula for $q(n)=\chi(n)(q)$:

Proposition 9.1. *For $x(m)\in\hat{\mathfrak g}$ and homogeneous $q\in V\otimes V$ we have*

$$[x(m),q(n)]=\sum_{i\geq 0}\binom{m}{i}(x_iq)(m+n),\qquad (Dq)(n)=-(n+wt\,q)q(n).$$

So if a subspace $Q\subset V\otimes V$ is invariant for $\tilde{\mathfrak g}_{\geq 0}$, then

$$\coprod_{n\in\mathbb{Z}}Q(n)$$

is a loop $\hat{\mathfrak g}$-module, in general reducible.

Now assume that $q=\sum a\otimes b$ is a homogeneous element in $\bar R\mathbf{1}\otimes V$. Note that for $a\in\bar R\mathbf{1}$ the coefficient $a(i)$ of the corresponding field $Y(a,z)$ can be written as a finite linear combination of basis elements $r(\rho)$, $\rho\in\ell t(\bar R)$. Hence each element of the sequence $q(n)=\chi(n)(\sum a\otimes b)\in\big(U\bar\otimes U\big)(n)$, say c_i, can be written uniquely as a finite sum of the form

$$c_i=\sum_{\rho\in\ell t(\bar R)}r(\rho)\otimes b_\rho,$$

where $b_\rho\in U$. If $b_\rho\neq 0$, then it is clear that $|\rho|+|\ell t(b_\rho)|=n$. Let us assume that $q(n)\neq 0$, and for nonzero "i-th" component c_i let π_i be the smallest possible $\rho\,\ell t(b_\rho)$ that appears in the expression for c_i. Denote by S the set of all such π_i. Since q is a finite sum of elements of the form $a\otimes b$, it is clear that there is ℓ such that $\ell(\pi_i)\leq\ell$. Then, by our assumptions on the order $\preceq$, the set S has the minimal element, and we call it the leading term $\ell t\big(q(n)\big)$ of $q(n)$. For a subspace $Q \subset \bar{R}\mathbf{1} \otimes V$ set

$$\ell t(Q(n)) = \{\ell t(q(n)) \mid q \in Q,\, q(n) \neq 0\}.$$

For a colored partition π of set

$$N(\pi) = \max\{\#\mathcal{E}(\pi){-}1,\, 0\}, \quad \mathcal{E}(\pi) = \{\rho \in \ell t(\bar{R}) \mid \rho \subset \pi\}.$$

Note that $V \otimes V$ has a natural filtration $(V \otimes V)_\ell$, $\ell \in \mathbb{Z}_{\geq 0}$, inherited from the filtration $\mathcal{U}_\ell$, $\ell \in \mathbb{Z}_{\geq 0}$. Then we have the following "rank theorem":

Theorem 9.2. *Let $Q \subset \ker\big(\Phi \mid (\bar{R}\mathbf{1} \otimes V)_\ell\big)$ be a finite-dimensional subspace and $n \in \mathbb{Z}$. Assume that $\ell(\pi) = \ell$ for all $\pi \in \ell t(Q(n))$. If*

$$\sum_{\pi \in \mathcal{P}^\ell(n)} N(\pi) = \dim Q(n), \tag{9.1}$$

then for any two embeddings $\rho_1 \subset \pi$ and $\rho_2 \subset \pi$ in $\pi \in \mathcal{P}^\ell(n)$, where $\rho_1, \rho_2 \in \ell t(\bar{R})$, we have a relation

$$u(\rho_1 \subset \pi) \in u(\rho_2 \subset \pi) + \overline{\mathbb{C}\text{-}span\{u(\rho \subset \pi') \mid \rho \in \ell t(\bar{R}), \rho \subset \pi', \pi \prec \pi'\}}. \tag{9.2}$$

Combinatorial relations (9.2) for the defining relations $r(\rho)$ of standard modules are needed for construction of combinatorial bases of standard modules. The left hand side of (9.1) is, for a given degree n, the total number $N(n)$ of relations needed, and the right hand side of (9.1) is the number of relations that we can construct by using the representation theory. As expected, $N(n) \geq \dim Q(n)$.

It should be noted that relations of the form (9.2) are easy to obtain when $\rho_1\rho_2 \subset \pi$. The problem is when two embeddings "intersect". Such relations for $r(\rho)$ of the combinatorial form (9.2) are obtained as linear combinations of relations constructed from "coefficients $q(n)$ of vertex operators $Y(q,z)$". In another words, a relation of the form (9.2) is a solution of certain system of linear equations, its existence is guaranteed by the condition (9.1).

10. The problem of constructing a combinatorial basis of $L(k\Lambda_0)$

We shall illustrate the (desired) construction of combinatorial bases of standard modules on the simpler case of $L(k\Lambda_0)$.

We assume we have an ordered basis B and we define the order $\preceq$ on $\mathcal{P}$ by comparing partitions gradually

(1) by length,
(2) by degree,
(3) by shape with reverse lexicographical order,
(4) by colors with reverse lexicographical order.

Set $r_{(k+1)\theta} = x_\theta(-1)^{k+1}\mathbf{1}$. Then, as in [MP1], we have

$$\ell t\left(r_{(k+1)\theta}(n)\right) = x_\theta(-j-1)^a x_\theta(-j)^b$$

with $a+b = k+1$ and $(-j-1)a+(-j)b = n$. Since we can obtain all other elements $r(n)$ for $r \in R$ by the adjoint action of $\mathfrak{g}$, which does not change the length and degree, we have that shapes of leading terms of $r(n)$ remain the same:

$$\operatorname{sh}\ell t\big(r(n)\big) = (-j-1)^a(-j)^b \tag{10.1}$$

with $a+b = k+1$ and $(-j-1)a+(-j)b = n$. Let us introduce the notation

$$\mathcal{D} = \ell t(\bar{R}) \cap \mathcal{P}_{<0}\,, \qquad \mathcal{RR} = P_{<0} \backslash \big(\mathcal{D} \cdot P_{<0}\big).$$

We shall denote by $\mathbf{1}$ the highest weight vector in the standard module $L(k\Lambda_0) = N(k\Lambda_0)/N^1(k\Lambda_0)$.

Proposition 10.1. *If for each $\ell \in \{k+2, \dots, 2k+1\}$ there exists a finite-dimensional subspace $Q_\ell \subset \ker\big(\Phi \,|\, (\bar{R}\mathbf{1} \otimes V)_\ell\big)$ such that $\ell(\pi) = \ell$ for all $\pi \in \ell t(Q_\ell(n))$ and*

$$\sum_{\pi \in \mathcal{P}^\ell(n)} N(\pi) = \dim Q_\ell(n),$$

for all $n \le -k-2$, then the set of vectors

$$u(\pi)\mathbf{1}\,, \qquad \pi \in \mathcal{RR}, \tag{10.2}$$

is a basis of the standard module $L(k\Lambda_0)$.

Proof. Since elements in R are of degree $k+1$, and there is no element in $N^1(k\Lambda_0)$ of smaller degree, for $\rho \in \ell t(\bar{R})$ we have that $r(\rho)\mathbf{1} = 0$ whenever $|\rho| > -k-1$. Hence (10.1) implies that $\rho \in \mathcal{D}$ whenever $r(\rho)\mathbf{1} \ne 0$. Since $N^1(k\Lambda_0) = \bar{R}N(k\Lambda_0) = U(\tilde{\mathfrak{g}}_{<0})\bar{R}\mathbf{1}$, we have a spanning set of $N^1(k\Lambda_0)$

$$u(\kappa)r(\rho)\mathbf{1} = u(\rho \subset \kappa\rho)\mathbf{1}\,, \qquad \kappa \in \mathcal{P}_{<0}\,,\ \rho \in \mathcal{D}.$$

For each $\pi \in \mathcal{D} \cdot P_{<0}$ choose exactly one $\rho_\pi \in \mathcal{D}$ such that $\rho_\pi \subset \pi$. Since by our assumptions we can apply Theorem 9.2, for each $\pi \in \mathcal{D} \cdot P_{<0}$ such that $\pi = \kappa_1\rho_1 = \kappa_2\rho_2$ we have a relation (9.2). Hence, by using induction, we se that

$$u(\rho_\pi \subset \pi)\mathbf{1}\,, \qquad \pi \in \mathcal{D} \cdot P_{<0}\,, \tag{10.3}$$

is a spanning set of $N^1(k\Lambda_0)$. Since by Proposition 8.5

$$\ell t\big(u(\pi/\rho_\pi)r(\rho_\pi)\big) = \big(\pi/\rho_\pi\big) \cdot \rho_\pi = \pi\,,$$

we have that

$$u(\rho_\pi \subset \pi)\mathbf{1} \in u(\pi)\mathbf{1} + U^{\mathcal{P}}_{(\pi)}\,\mathbf{1},$$

and by induction we see that the set (10.3) is linearly independent. Hence this set is a basis of $N^1(k\Lambda_0)$.

In the obvious way we can assign to each colored partition π its weight $\mathrm{wt}\,\pi$, and we have characters

$$\mathrm{ch}\,N(k\Lambda_0) = \sum_{\pi \in \mathcal{P}_{<0}} e^{\mathrm{wt}\,\pi}, \qquad \mathrm{ch}\,N^1(k\Lambda_0) = \sum_{\pi \in \mathcal{D}\cdot P_{<0}} e^{\mathrm{wt}\,\pi}.$$

Hence we have

$$\mathrm{ch}\,L(k\Lambda_0) = \sum_{\pi \in \mathcal{RR}} e^{\mathrm{wt}\,\pi}. \tag{10.4}$$

To find a basis of $L(k\Lambda_0)$ we start with the PBW spanning set

$$u(\pi)\mathbf{1}\,, \qquad \pi \in \mathcal{P}_{<0}\,.$$

For $\pi \in \mathcal{D} \cdot P_{<0}$ we have

$$u(\pi) \in u(\pi/\rho_\pi)r(\rho_\pi) + U^{\mathcal{P}}_{(\pi)}.$$

Since $r(\rho_\pi)\mathbf{1} = 0$ in $L(k\Lambda_0)$, we have

$$u(\pi)\mathbf{1} \in U^{\mathcal{P}}_{(\pi)}\,\mathbf{1} \quad \text{for } \pi \in \mathcal{D} \cdot P_{<0},$$

and by using induction we can reduce the PBW spanning set to a spanning set (10.2). By the character formula (10.4) this set is linearly independent. $\square$

Remarks. (i) At the moment just a few examples are known where the conditions of Theorem 9.2 are satisfied, the simplest is for the basic $\mathfrak{sl}(2,\mathbb{C})\tilde{}$-module (see [MP1]). With the usual notation $x = x_\theta$, $h = \theta^\vee$ and $y = x_{-\theta}$, the set of leading terms $\ell t(\bar{R})$ is:

$$b_1(-j)b_2(-j) \quad \text{with colors } b_1b_2: \quad yy, yh, hh, hx, xx,$$

$$b_1(-j-1)b_2(-j) \quad \text{with colors } b_1b_2: \quad yy, hy, xy, xh, xx.$$

If one takes

$$Q_3 = U(\mathfrak{g})q_{2\theta} \oplus U(\mathfrak{g})q_{3\theta},$$

then, by using Proposition 9.1 and loop modules, $\dim Q_3(n) = 5+7$ and (9.1) holds for all n. If one takes $(1,2)$-specialization of the Weyl-Kac character formula on one side, and (10.4) on the other side, one obtains a Capparelli identity [C].

(ii) The results in Section 9 can be extended to twisted affine Lie algebras (see [P1]). In such formulation of Theorem 9.2 the equality (9.1) also holds for level 1 twisted $\mathfrak{sl}(3,\mathbb{C})\tilde{}$-modules (see [S]).

(iii) The character formula (10.4) is a generating function for numbers of colored partitions in $\mathcal{RR}$ satisfying "difference $\mathcal{D}$ conditions", and combined with the Weyl-Kac character formula gives a Rogers-Ramanujan type identity.

11. Combinatorial bases of basic modules for $C_n^{(1)}$

We fix a simple Lie algebra $\mathfrak{g}$ of type C_n, $n \geq 2$. For a given Cartan subalgebra $\mathfrak{h}$ and the corresponding root system Δ we can write

$$\Delta = \{\pm(\varepsilon_i \pm \varepsilon_j) \mid i,j = 1,...,n\} \setminus \{0\} .$$

We chose simple roots as in [Bou]

$$\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \alpha_2 = \varepsilon_2 - \varepsilon_3,\ \cdots\ \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n,\ \alpha_n = 2\varepsilon_n.$$

Then $\theta = 2\varepsilon_1$ and $\alpha^\star = \alpha_1$. By Lemma 7.5 for each degree m we have a space of relations for annihilating fields

$$Q_3(m) = U(\mathfrak{g})q_{2\theta}(m) \oplus U(\mathfrak{g})q_{3\theta}(m) \oplus U(\mathfrak{g})q_{3\theta-\alpha^*}(m) .$$

The Weyl dimension formula for $\mathfrak{g}$ gives

$$\dim L(2\theta) = \binom{2n+3}{4}, \tag{11.1}$$

$$\dim L(3\theta) = \binom{2n+5}{6}, \tag{11.2}$$

$$\dim L(3\theta-\alpha^\star) = \frac{(2n+5)(n-1)}{3}\binom{2n+3}{4}. \tag{11.3}$$

Hence we have

$$\dim Q_3(m) = \dim L(2\theta) + \dim L(3\theta) + \dim L(3\theta-\alpha^\star) = 2n\binom{2n+4}{5}. \tag{11.4}$$

For each $\alpha \in \Delta$ we chose a root vector x_α such that $[x_\alpha, x_{-\alpha}] = \alpha^\vee$. For root vectors x_α we shall use the following notation:

$$\begin{array}{llll} x_{ij} & \text{or just } ij & \text{if} & \alpha = \varepsilon_i + \varepsilon_j\ ,\ i \leq j\ , \\ x_{\underline{i}\underline{j}} & \text{or just } \underline{i}\underline{j} & \text{if} & \alpha = -\varepsilon_i - \varepsilon_j\ ,\ i \geq j\ , \\ x_{i\underline{j}} & \text{or just } i\underline{j} & \text{if} & \alpha = \varepsilon_i - \varepsilon_j\ ,\ i \neq j\ . \end{array}$$

With previous notation $x_\theta = x_{11}$. We also write for $i = 1, \dots, n$

$$x_{i\underline{i}} = \alpha_i^\vee \text{ or just } i\underline{i}\,.$$

These vectors x_{ab} form a basis B of $\mathfrak{g}$ which we shall write in a triangular scheme. For example, for $n = 3$ the basis B is

$$\begin{array}{llllll}
11 & & & & & \\
12 & 22 & & & & \\
13 & 23 & 33 & & & \\
1\underline{3} & 2\underline{3} & 3\underline{3} & \underline{33} & & \\
1\underline{2} & 2\underline{2} & 3\underline{2} & \underline{32} & \underline{22} & \\
1\underline{1} & 2\underline{1} & 3\underline{1} & \underline{31} & \underline{21} & \underline{11}.
\end{array}$$

In general for the set of indices $\{1, 2, \cdots, n, \underline{n}, \cdots, \underline{2}, \underline{1}\}$ we use order

$$1 \succ 2 \succ \cdots \succ n-1 \succ n \succ \underline{n} \succ \underline{n-1} \succ \cdots \succ \underline{2} \succ \underline{1}$$

and a basis element x_{ab} we write in a^{th} column and b^{th} row,

$$B = \{x_{ab} \mid b \in \{1, 2, \cdots, n, \underline{n}, \cdots, \underline{2}, \underline{1}\},\ a \in \{1, \cdots, b\}\}. \tag{11.5}$$

By using (11.5) we define on B the corresponding reverse lexicographical order, i.e.

$$x_{ab} \succ x_{a'b'} \ \mathit{if}\ b \succ b' \ \mathit{or}\ b = b' \ \mathit{and}\ a \succ a'\ . \tag{11.6}$$

In other words, x_{ab} is larger than $x_{a'b'}$ if $x_{a'b'}$ lies in a row b' below the row b, or x_{ab} and $x_{a'b'}$ are in the same row $b = b'$, but $x_{a'b'}$ is to the right of x_{ab}.

For $r \in \{1, \cdots, n, \underline{n}, \cdots, \underline{1}\}$ we introduce the notation

$$\triangle_r \quad \text{and} \quad {}^r\triangle$$

for triangles in B consiting of rows $\{1, \dots, r\}$ and columns $\{r, \dots, \underline{1}\}$. For example, for $n = 3$ and $r = \underline{3}$ we have triangles $\triangle_{\underline{3}}$ and ${}^{\underline{3}}\triangle$

$$\begin{array}{llllll}
11 & & & & & \\
12 & 22 & & & & \\
13 & 23 & 33 & & & \\
1\underline{3} & 2\underline{3} & 3\underline{3} & \underline{33} & & \\
 & & & \underline{32} & \underline{22} & \\
 & & & \underline{31} & \underline{21} & \underline{11}.
\end{array}$$

With order $\preceq$ on B we define a linear order on $\bar{B} = \{x(j) \mid x \in B, j \in \mathbb{Z}\}$ by

$$x_\alpha(i) \prec x_\beta(j) \quad \text{if} \quad i < j \quad \text{or} \quad i = j,\ x_\alpha \prec x_\beta. \tag{11.7}$$

With order $\preceq$ on $\bar{B}$ we define a linear order on $\mathcal{P}$ by

$$\pi \prec \kappa \quad \text{if}$$

- $\ell(\pi) > \ell(\kappa)$ or
- $\ell(\pi) = \ell(\kappa)$, $|\pi| < |\kappa|$ or
- $\ell(\pi) = \ell(\kappa)$, $|\pi| = |\kappa|$, $\operatorname{sh}\pi \prec \operatorname{sh}\kappa$ in the reverse lexicographical order or
- $\ell(\pi) = \ell(\kappa)$, $|\pi| = |\kappa|$, $\operatorname{sh}\pi = \operatorname{sh}\kappa$ and colors of π are smaller than the colors of κ in reverse lexicographical order.

Lemma 11.1. *The set of leading terms of relations $\bar{R}$ for level 1 standard $\tilde{\mathfrak{g}}$-modules consists of quadratic monomials*

$$x_{a_1b_1}(-j)x_{a_2b_2}(-j), \quad j \in \mathbb{Z}, \quad b_2 \preceq b_1 \ \mathit{and}\ a_2 \succeq a_1,$$

and quadratic monomials

$$x_{a_1b_1}(-j-1)x_{a_2b_2}(-j), \quad j \in \mathbb{Z}, \quad b_1 \succeq a_2.$$

This lemma is a special case of Theorem 6.1 in [PŠ]. The proof for this level one case reduces to a very simple argument.

Remark 11.2. Note that a quadratic monomial $x_{a_1b_1}(-j-1)x_{a_2b_2}(-j)$ is a leading term of relation if and only if there is r such that

$$x_{a_1b_1} \in \triangle_r \quad \text{and} \quad x_{a_2b_2} \in {}^r\triangle.$$

Theorem 11.3. *The set of monomial vectors which have no leading term as a factor, i.e., the set of vectors*

$$u(\pi)\mathbf{1}\,, \qquad \pi \in \mathcal{RR}, \tag{11.8}$$

is a basis of the basic $\tilde{\mathfrak{g}}$*-module* $L(\Lambda_0)$.

Proof. By Proposition 10.1 and (11.4) it is enough to show

$$\sum_{\pi \in \mathcal{P}^3(m)} N(\pi) = 2n\binom{2n+4}{5}\,. \tag{11.9}$$

In order to simplify the counting of embeddings of leading terms we introduce a slightly different indexation of a triangular scheme for a basis B. By using

$$k \mapsto k \quad \underline{k} \mapsto 2n-k+1 \tag{11.10}$$

and matrix notation for rows and columns we can rewrite the basis

$$B = \{x_{k,l} \mid k \in \{1,\cdots,2n\}\,,\ l \in \{1,\cdots,k\}\}\,.$$

We need to count embeddings in (11.9) for $m = -3j-1$, $-3j-2$ and $-3j-3$, that is, we need to consider three cases:

(I) $x_{k_1l_1}(-j-1)x_{k_2l_2}(-j)x_{k_3l_3}(-j)$ where $x_{k_2l_2} \preceq x_{k_3l_3}$
(II) $x_{k_1l_1}(-j-1)x_{k_2l_2}(-j-1)x_{k_3l_3}(-j)$ where $x_{k_1l_1} \preceq x_{k_2l_2}$
(IIIa) $x_{k_1l_1}(-j-2)x_{k_2l_2}(-j-1)x_{k_3l_3}(-j)$
(IIIb) $x_{k_1l_1}(-j-1)x_{k_2l_2}(-j-1)x_{k_3l_3}(-j-1)$ where $x_{k_1l_1} \preceq x_{k_2l_2} \preceq x_{k_3l_3}$.

Denote by N the number of embeddings. During counting embeddings of leading terms we need to multiply the count by a factor $N-1$. We describe calculation of the first case in all details.

The first case $x_{k_1l_1}(-j-1)x_{k_2l_2}(-j)x_{k_3l_3}(-j)$ where $x_{k_2l_2} \preceq x_{k_3l_3}$.
Depending on the type and number of embeddings the first case is split in the following five subcases:

(I1) $N=3$; $x_{k_1l_1}(-j-1)x_{k_2l_2}(-j)$, $x_{k_1l_1}(-j-1)x_{k_3l_3}(-j)$ and $x_{k_2l_2}(-j)x_{k_3l_3}(-j)$ are leading terms (+ condition $x_{k_2l_2} \neq x_{k_3l_3}$)
(I2) $N=2$; $x_{k_1l_1}(-j-1)x_{k_2l_2}(-j)$, $x_{k_1l_1}(-j-1)x_{k_3l_3}(-j)$ and $x_{k_2l_2}(-j)x_{k_3l_3}(-j)$ are leading term (+ condition $x_{k_2l_2} = x_{k_3l_3}$)
(I3) $N=2$; $x_{k_1l_1}(-j-1)x_{k_2l_2}(-j)$, $x_{k_1l_1}(-j-1)x_{k_3l_3}(-j)$ are leading terms and $x_{k_2l_2}(-j)x_{k_3l_3}(-j)$ not leading terms (+ condition $x_{k_2l_2} \neq x_{k_3l_3}$)
(I4) $N=2$; $x_{k_1l_1}(-j-1)x_{k_2l_2}(-j)$ not leading term, $x_{k_1l_1}(-j-1)x_{k_3l_3}(-j)$ and $x_{k_2l_2}(-j)x_{k_3l_3}(-j)$ are leading terms (+ condition $x_{k_2l_2} \neq x_{k_3l_3}$)
(I5) $N=2$; $x_{k_1l_1}(-j-1)x_{k_2l_2}(-j)$ is leading term, $x_{k_1l_1}(-j-1)x_{k_3l_3}(-j)$ not leading term and $x_{k_2l_2}(-j)x_{k_3l_3}(-j)$ is leading term (+ condition $x_{k_2l_2} \neq x_{k_3l_3}$)

Subcase (I1): Note that $x_{k_1 l_1}(-j-1) \in \triangle_r$ and $x_{k_2 l_2}(-j) \in {}^r\triangle$ and selection of their position is entirely free. Therefore, the number of embedded leading terms in this subcase is given by

$$\sum_{I1} = \sum_{k_1=1}^{2n} \sum_{l_1=1}^{k_1} \sum_{k_2=k_1}^{2n} \sum_{l_2=k_1}^{k_2} [(N-1)(\sharp\ x_{k_3 l_3})] \tag{11.11}$$

where $\sharp\ x_{k_3 l_3}$ is number of admissible position for $x_{k_3 l_3}$. Since $x_{k_2 l_2} \preceq x_{k_3 l_3}$ then $\sharp\ x_{k_3 l_3} = (l_2 - k_1) + [1 + 2 + \cdots + (k_2 - l_2)]$ (see Figure 1) and the sum (11.11) is

$$\sum_{I1} = \sum_{k_1=1}^{2n} \sum_{l_1=1}^{k_1} \sum_{k_2=k_1}^{2n} \sum_{l_2=k_1}^{k_2} [2(l_2 - k_1) + (k_2 - l_2)(k_2 - l_2 + 1)] \tag{11.12}$$

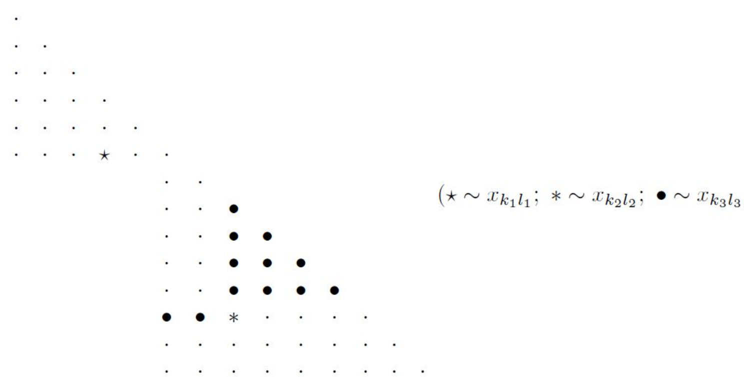


Figure 1. Subcase (I1)

Subcase (I2): This subcase is similar as subcase (I1) for $N = 2$ and $\sharp\ x_{k_3 l_3} = 1$. From this immediately follows

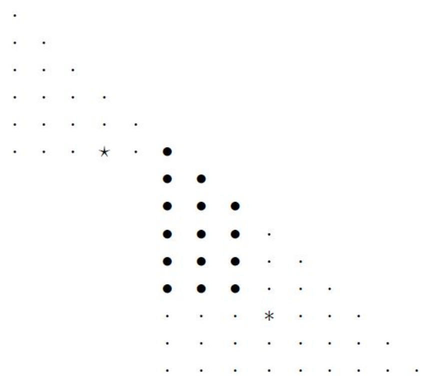
$$\sum_{I2} = \sum_{k_1=1}^{2n} \sum_{l_1=1}^{k_1} \sum_{k_2=k_1}^{2n} \sum_{l_2=k_1}^{k_2} 1 \ . \tag{11.13}$$

Subcase (I3): In this subcase we have again the same following setting

$$N = 2\ ; x_{k_2 l_2} \prec x_{k_3 l_3}\ ;\ x_{k_1 l_1}(-j-1) \in \triangle_r\ ;\ x_{k_2 l_2}(-j) \in {}^r\triangle\ .$$

Since the $x_{k_2 l_2}(-j) x_{k_3 l_3}(-j)$ is not the leading term then $\sharp\ x_{k_3 l_3} = \frac{(l_2 - k_1)(2k_2 - k_1 - l_2 + 1)}{2}$ (see Figure 2) and the sum $\sum_{I3}$ is

$$\sum_{I3} = \sum_{k_1=1}^{2n} \sum_{l_1=1}^{k_1} \sum_{k_2=k_1}^{2n} \sum_{l_2=k_1}^{k_2} [\frac{(l_2 - k_1)(2k_2 - k_1 - l_2 + 1)}{2}] \ . \tag{11.14}$$



$(\star \sim x_{k_1l_1};\ * \sim x_{k_2l_2};\ \bullet \sim x_{k_3l_3})$

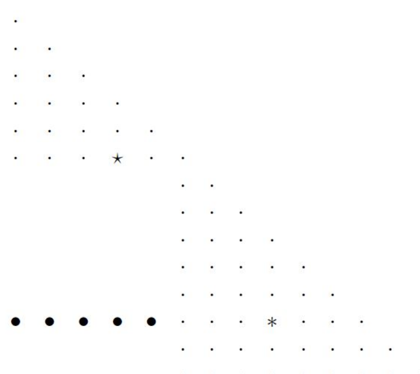
FIGURE 2. Subcase (I3)

Subcase (I4): In this subcase we have the following setting

$$N = 2\ ; x_{k_2l_2} \prec x_{k_3l_3}\ ;\ x_{k_1l_1}(-j-1) \in \triangle_r\ ;\ x_{k_2l_2}(-j) \in {}^r\triangle\ .$$

Since the $x_{k_1l_1}(-j-1)x_{k_3l_3}(-j)$ is not the leading term then $\sharp\ x_{k_3l_3} = k_1 - 1$ (see Figure 3) and the sum $\sum_{I4}$ is

$$\sum_{I4} = \sum_{i_1=1}^{2n} \sum_{j_1=1}^{i_1} \sum_{i_2=i_1}^{2n} \sum_{j_2=i_1}^{i_2} [i_1 - 1]\ . \tag{11.15}$$



$(\star \sim x_{k_1l_1};\ * \sim x_{k_2l_2};\ \bullet \sim x_{k_3l_3})$

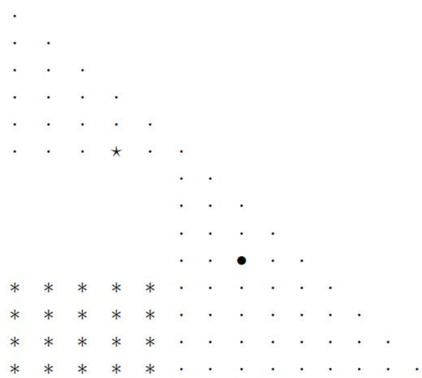
FIGURE 3. Subcase (I4)

Subcase (I5): Since in this subcase $x_{k_1l_1}(-j-1)x_{k_2l_2}(-j)$ is not the leading term then we select entirely free the position of $x_{k_1l_1}(-j-1) \in \triangle_r$ and $x_{k_3l_3}(-j) \in {}^r\triangle$. Then the corresponding setting is

$$N = 2\ ; x_{k_2l_2} \prec x_{k_3l_3}\ ;\ x_{k_1l_1}(-j-1) \in \triangle_r\ ;\ x_{k_3l_3}(-j) \in {}^r\triangle\ .$$

Since the $x_{k_1 l_1}(-j-1) x_{k_2 l_2}(-j)$ is not the leading term then $\sharp\, x_{k_2 l_2} = (2n-k_3)(k_1-1)$ (see Figure 4) and the sum $\sum_{I5}$ is

$$\sum_{I5} = \sum_{k_1=1}^{2n} \sum_{l_1=1}^{k_1} \sum_{k_3=k_1}^{2n} \sum_{l_3=k_1}^{k_3} (2n-k_3)(k_1-1)\ . \tag{11.16}$$



$(\star \sim x_{k_1 l_1};\ * \sim x_{k_2 l_2};\ \bullet \sim x_{k_3 l_3})$

FIGURE 4. Subcase (I5)

Finally we have

$$\sum_{I1} + \sum_{I2} + \sum_{I3} + \sum_{I4} + \sum_{I5} = 2n \binom{2n+4}{5}\ .$$

In other two cases counting of embeddings of the leading terms is similar and shows that (11.9) holds. $\square$

12. Combinatorial Rogers-Ramanujan type identities

As a consequence of Theorem 11.3 we have a combinatorial Rogers-Ramanujan type identities by using Lepowsky's product formula for principaly specialized characters of $C_n^{(1)}$-modules $L(\Lambda_0)$ (see [L] and [M], cf. [MP2] for $n=1$)

$$\prod_{\substack{j\geq 1\\ j\not\equiv 0 \bmod 2}} \frac{1}{1-q^j} \qquad \prod_{\substack{j\geq 1\\ j\not\equiv 0,\pm 1 \bmod n+2}} \frac{1}{1-q^{2j}}. \tag{12.1}$$

This product can be interpreted combinatorially as a generating function for number of partitions

$$N = \sum_{m\geq 1} m f_m. \tag{12.2}$$

of N with parts m satisfying congruence conditions.

$$f_m = 0 \quad \text{if} \quad m \equiv 0, \pm 2 \bmod 2n+4. \tag{12.3}$$

On the other hand, in the principal specialization $e^{-\alpha_i} \mapsto q^1$, $i = 0, 1, \ldots, n$, the sequence of basis elements in $C_n^{(1)}$

$$X_{ab}(-1),\ ab \in B, \quad X_{ab}(-2),\ ab \in B, \quad X_{ab}(-3),\ ab \in B, \quad \ldots \tag{12.4}$$

obtains degrees

$$|X_{ab}(-j)| = a + b - 1 + 2n(j-1),$$

where we prefer row and column indices of basis elements $X_{ab} \in B$ to be natural numbers

$$b = 1, \dots, 2n, \qquad a = 1, \dots, b.$$

For example, the basis elements for $C_2^{(1)}$ in the sequence (12.4) obtan degrees

$$\begin{array}{cccccccccccccc} 1 & & & & 5 & & & & 9 & & & & \\ 2 & 3 & & & 6 & 7 & & & 10 & 11 & & & \\ 3 & 4 & 5 & & 7 & 8 & 9 & & 11 & 12 & 13 & & \dots \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \end{array} \tag{12.5}$$

As we see, there are several basis elements of a given degree m,

$$m = a + b - 1 + 2n(j-1),$$

so we make them "distinct" by assigning to each degree m a "color" b, the row index in which m appears:

$$m_b, \qquad |m_b| = m.$$

For example, for $n = 2$ we have

$$\begin{array}{cccccccccccccc} 1_1 & & & & 5_1 & & & & 9_1 & & & & \\ 2_2 & 3_2 & & & 6_2 & 7_2 & & & 10_2 & 11_2 & & & \\ 3_3 & 4_3 & 5_3 & & 7_3 & 8_3 & 9_3 & & 11_3 & 12_3 & 13_3 & & \dots, \\ 4_4 & 5_4 & 6_4 & 7_4 & 8_4 & 9_4 & 10_4 & 11_4 & 12_4 & 13_4 & 14_4 & 15_4 & \end{array} \tag{12.6}$$

so that numbers in the first row have color 1, numbers in the second row have color 2, and so on. In general we consider a disjoint union $\mathcal{D}_n$ of integers in $2n$ colors, say $m_1, m_2, \dots, m_{2m}$, satisfying the congruence conditions

$$\begin{aligned} &\{m_1 \mid m \geq 1, m \equiv 1 \bmod 2n\}, \\ &\{m_2 \mid m \geq 2, m \equiv 2, 3 \bmod 2n\}, \\ &\{m_3 \mid m \geq 3, m \equiv 3, 4, 5 \bmod 2n\}, \\ &\quad \dots \\ &\{m_{2n} \mid m \geq 2n, m \equiv 2n, 2n+1, \dots, 4n-1 \bmod 2n\} \end{aligned} \tag{12.7}$$

and arranged in a sequence of triangles.

For fixed m and b parameters a and j are completely determined. We see this easily for the last row

$$2n_{2n}, \dots, (4n-1)_{2n}; 4n_{2n}, \dots, (6n-1)_{2n}; 6n_{2n}, \dots\ ,$$

and then for all the other rows as well. So instead of m_b we may write $m_{ab}(-j)$.

Theorem 12.1. *For every positive integer N the number of partitions*

$$N = \sum_{m \geq 1} m f_m$$

with congruence conditions $f_m = 0$ if $m \equiv 0, \pm 2 \bmod 2n+4$ equals the number of colored partitions

$$N = \sum_{m_b \in \mathcal{D}_n} |m_b| f_{m_b} \tag{12.8}$$

with difference conditions $f_{m_b} + f_{m'_{b'}} \leq 1$ if

- *$m_b = m_{ab}(-j-1)$ and $m'_b = m'_{a'b'}(-j)$ such that $b \geq a'$, or*
- *$m_b = m_{ab}(-j)$ and $m'_b = m'_{a'b'}(-j)$ such that $b \leq b'$, $a \geq a'$.*

For adjacent triangles corresponding to

$$\ldots, \quad X_{ab}(-j), ab \in B, \quad X_{ab}(-j-1), ab \in B, \quad \ldots$$

in (12.4) and a fixed row $r \in \{1, \ldots, 2n\}$ we consider the corresponding two triangles: ${}^r\triangle$ on the left and $\triangle_r$ on the right. For example, for $n = 2$ and the third row we have $r = 3$ and two triangles denoted by bullets:

$$\begin{array}{cccccccccccccccccc}
 & \cdot & & & & \cdot & & & & \bullet & & & & \cdot & & & & \\
 & \cdot & \cdot & & & \cdot & \cdot & & & \bullet & \bullet & & & \cdot & \cdot & & & \\
\cdots & \cdot & \cdot & \cdot & & \cdot & \cdot & \bullet & & \bullet & \bullet & \bullet & & \cdot & \cdot & \cdot & & \cdots \\
 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot &
\end{array} \tag{12.9}$$

are ${}^3\triangle$ on the left and $\triangle_3$ on the right.

Then the first difference condition **does not allow** two parts in a colored partition (12.8) such that

$$m'_b = m'_{a'b'}(-j) \in {}^r\triangle \quad \text{and} \quad m_b = m_{ab}(-j-1) \in \triangle_r.$$

On the other hand, the second difference condition **does not allow** two parts in a colored partition (12.8) such that

$$m'_b = m'_{a'b'}(-j), \quad m_b = m_{ab}(-j)$$

to be in any rectangle such as:

$$\begin{array}{cccccccccccccccccc}
 & \cdot & & & & \cdot & & & & \cdot & & & & \cdot & & & & \\
 & \cdot & \cdot & & & \cdot & \cdot & & & \cdot & \cdot & & & \cdot & \cdot & & & \\
\cdots & \cdot & \cdot & \cdot & & \bullet & \bullet & \bullet & & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & \cdots \\
 & \cdot & \cdot & \cdot & \cdot & \bullet & \bullet & \bullet & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot &
\end{array} \tag{12.10}$$

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