Analytical procedures for torsional vibration analysis of ship power transmission system

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Abstract

In this paper two relatively simple analytical procedures for free and forced torsional vibration analysis of ship power transmission systems are developed. In the first, approximate procedure, the shaft line is modelled as a two-mass system and analytical solution of the differential equations of motion is given. In the second one, a multi degree of freedom (d.o.f.) problem of the complete propulsion system is solved by the Rayleigh-Ritz method. A special attention is paid to the determination of the contribution of each cylinder to the primary and secondary engine torques by taking into account the firing order. The application of the two procedures is illustrated in the case of a typical propulsion system of a merchant ship with a slow-speed main engine connected directly to the propeller by a relatively short shaft line. The obtained results are verified by a comparison with measurements. All classification societies require calculation of the propulsion system operating parameters, but they do not provide simplified formulae for vibration analysis. The outlined analytical procedures can be used for the estimation of torsional vibration of the shaft line in the preliminary ship design stage as well as for ships in service.

Keywords: Power transmission system, propulsion system, shaft line, engine excitation, torsional vibration, analytical procedure

1. Introduction

One of the most important prerequisites for safe navigation of ships at sea is a reliable design of the propulsion system and an efficient control of the corresponding vibration, [1]. Since the oil crisis in 1973, new types of long-stroke and slow-speed diesel engines with a small number of cylinders have emerged as the main propulsion system in merchant ships, [2, 3]. Although these engines are characterised by a remarkable fuel efficiency, the consequent high power output delivered per one cylinder has increased the amplitudes of the excitation forces that induce vibrations of the ship power transmission line. The frequency spectra of these forces overlap with natural frequencies of the main engine and the ship structure. This usually results in excessive resonant vibrations, which have a dangerous influence on the crew health and comfort, ship equipment and ship safety in general. Therefore, the examination of vibration characteristics and the vibration control in complex ship structures has become an imperative, [4-6]. A number of remedial solutions and systems for the control of engine induced vibration in ships have been developed, such as, for example, the installation of engine stays, [2, 3].

Given the fact that the propulsion system is one of the main sources of ship vibrations, in the middle of the last century the leading classification societies specified in their rules simple semi-empirical formulae for the estimation of propulsion system vibrations. By developing digital computers and numerical methods, these formulae have been replaced with guidelines for the so-called "direct calculation" of shaft line torsional vibrations, [7], [8], [9]. Nowadays, there are several software packages mostly based on the Finite Element Method (FEM), [10], applicable for this purpose. However, these tools are complex and require specialized engineering skills.

Torsional vibrations of the propulsion system are still among the most dangerous for the shaft line, [11]. This is especially true in the case of five-cylinder low-speed engines, [12]. If the diameter of the shaft line is chosen solely according to the Classification Society Rules, the resulting torsional vibration stresses may still be significantly above the permissible limits. This problem can be overcome in one of the following ways: the flexible shafting system design, the rigid shafting system design, and by mounting a torsional vibration damper, [13]. The first two solutions are quite common, whereas the third one is not.

In general, there are two approaches to analysing the shaft line vibrations. In the first and the more usual approach, the shaft line is treated separately from the rest of the ship for simplicity. Since the coupling between the shaft line vibrations and the vibrations of the ship hull is ignored, the problem of boundary conditions is avoided, [14]. In the other approach, the propulsion system is modelled together with either the aft part of the ship hull, or with the complete ship structure, [6, 10].

The shaft line consists of the crankshaft, the intermediate shaft, the propeller shaft, and the optional couplings and gears. Torsional vibrations of the shaft line are excited by the pulsating torque generated by the reciprocating combustion engine, [15], as well as by the propeller beats in the non-uniform wake field, [16]. The problem of the shaft line torsional vibrations has been investigated since the 1950s, [17,18]. In spite of such intensive research some issues still remain open. These include the determination of the propeller damping, the cylinder damping, and the added inertia of the surrounding water to the polar moment of inertia of the propeller, [14]. Also, axial vibrations and whirling vibrations of the propulsion system have been a subject of some recent investigations, [19, 20].

An effort to simplify the numerical procedure for the estimation of free and forced propulsion system vibrations has been undertaken in [14]. The shaft line is modelled as a two mass system. However, the procedure is based on some intuitively introduced assumptions concerning both the resulting engine torque of cylinder excitations and the transfer of engine excitation to the shaft response, which are not entirely physically based.

With regard to the state-of-the-art and the practical needs, two relatively simple analytical procedures for the estimation of torsional shaft line vibrations are described in this paper. In the first procedure, the propulsion system is condensed into a two d.o.f. system, encompassing the propeller mass and the crankshaft mass. The governing differential equations of motion are solved analytically. With the second procedure, the propulsion system is modelled as a multi d.o.f. system with lumped masses, ignoring the shaft inertia. The energy formulation is used, and the problem is solved by employing the Rayleigh-Ritz method.

A special attention is paid to the evaluation of the cylinder excitation and the resulting engine torque by taking into account the firing order. Two cases are distinguished, i.e. the primary and the secondary engine excitation, depending on whether or not the ordinary number of the excitation harmonics is equal to the number of engine cylinders, respectively. The accuracy of the presented procedures is verified by a comparison of the calculated intermediate shaft stresses with the measured values, [14].

2. Condensed model of propulsion system

2.1. Differential equations of motion

The shaft line consists of three main shafts, i.e. the propeller shaft, the intermediate shaft and the crankshaft, with the corresponding lumped masses of the propeller, the flywheel and the cranks of the crankshaft. Since the crankshaft is normally significantly stiffer than the propeller shaft or the intermediate shaft, the crankshaft masses and the flywheel mass can be condensed into one mass at the end of the intermediate shaft. Such a two d.o.f. model with nodal twist angles $\phi_0(t)$ and $\phi_1(t)$ is shown in Fig. 1, where J_0 is the propeller polar moment of inertia which includes the added polar moment of inertia due to the surrounding water, and J_1 is the polar moment of inertia of the condensed crankshaft mass.

The shaft is exposed to the action of the following torsional moments, Fig. 1.:

1) Engine excitation torque

$$M_E(t) = M_E \cos\Omega t\,, \tag{1}$$

where Ω is the engine angular speed and the fundamental excitation frequency.

2) Propeller inertia moment and crankshaft inertia moment

$$M_{i0}(t) = J_0\ddot{\phi}_0\,,\; M_{i1}(t) = J_1\ddot{\phi}_1\,. \tag{2}$$

3) Propeller damping moment and engine damping moment

$$M_{d0}(t) = c_0\dot{\phi}_0, M_{d1}(t) = c_1\dot{\phi}_1, \tag{3}$$

where c_0 and c_1 are the damping coefficients.

4) Internal torsional moments, as a result of shear stresses over the shaft cross-section area

$$M_0(t) = M_1(t) = GI\frac{\partial\phi}{\partial x} = K(\phi_1 - \phi_0)\,, \tag{4}$$

where K is the equivalent torsional stiffness of the propeller shaft and intermediate shaft

$$K = \frac{1}{\frac{1}{K_0} + \frac{1}{K_1}}\,,\; K_0 = \frac{GI_0}{L_0}\,,\; K_1 = \frac{GI_1}{L_1}\,. \tag{5}$$

The distribution of the approximate twist angle $\phi(t)$ and the cross-sectional torsional moment $M(t)$ along the shaft is shown in Fig. 1.

In order to enable simpler formulation of the equilibrium conditions, all the angular displacements and torsional moments in Fig. 1 are presented as vectors in Fig. 2, according to the left hand rule. It can be seen that the torsional shaft model is analogous to a bar tensional model. In order to formulate the dynamic equilibrium equations of the shaft, it is split into two parts, as shown in Fig. 2. The external and internal moments must be in equilibrium for each part of the shaft, i.e.

$$\begin{aligned} &M_0(t) - M_{d0}(t) - M_{i0}(t) = 0 \\ &M_1(t) + M_{d1}(t) + M_{i1}(t) = M_E(t). \end{aligned} \tag{6}$$

Substituting Eqs. (1) (2), (3) and (4) into (6) one obtains two differential equations of motion coupled by the stiffness K

$$\begin{aligned} &K(\phi_1 - \phi_0) - c_0\dot{\phi}_0 - J_0\,\ddot{\phi}_0 = 0 \\ &K(\phi_1 - \phi_0) + c_1\dot{\phi}_1 + J_1\,\ddot{\phi}_1 = M_E(t). \end{aligned} \tag{7}$$

The harmonic excitation torque (1) can be specified as the real part of the complex torque

$$\tilde{M}_E(t) = M_E \mathrm{e}^{i\Omega t}\,. \tag{8}$$

where $i = \sqrt{-1}$ is the imaginary unit. Accordingly, the complex twist angles are assumed in the same form

$$\tilde{\phi}_0(t) = \tilde{A}_0 \mathrm{e}^{i\Omega t}\,,\ \tilde{\phi}_1(t) = \tilde{A}_1 \mathrm{e}^{i\Omega t}\,, \tag{9}$$

where $\tilde{A}_0$ and $\tilde{A}_1$ are now complex constants.

The system of differential equations (7) in the complex domain reads

$$\begin{aligned} &\tilde{\phi}_1 - \tilde{\phi}_0 - \frac{c_0}{K}\dot{\tilde{\phi}}_0 - \frac{J_0}{K}\ddot{\tilde{\phi}}_0 = 0 \\ &\tilde{\phi}_1 - \tilde{\phi}_0 + \frac{c_1}{K}\dot{\tilde{\phi}}_1 + \frac{J_1}{K}\ddot{\tilde{\phi}}_1 = \frac{\tilde{M}_E(t)}{K}. \end{aligned} \tag{10}$$

Substituting Eqs. (8) and (9) into (10), yields

$$\begin{bmatrix} -\left(1 + i\Omega\dfrac{c}{K} - \Omega^2\dfrac{J_0}{K}\right) & 1 \\ -1 & \left(1 + i\Omega\dfrac{c_1}{K} - \Omega^2\dfrac{J_1}{K}\right) \end{bmatrix} \begin{Bmatrix} \tilde{A}_0 \\ \tilde{A}_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \dfrac{M_E}{K} \end{Bmatrix}. \tag{11}$$

2.2. Natural vibrations

In case of natural vibrations there is no excitation, $(M_E = 0)$. Furthermore, the influence of damping is considered to be negligible, $(c_0 = c_1 = 0)$. The two twists angles are assumed in the form

$$\phi_0 = \varphi_0 \cos\omega t\,,\ \phi_1 = \varphi_1 \cos\omega t\,, \tag{12}$$

where ω is a natural frequency. In this case Eq. (11) is reduced to the homogenous one

$$\begin{bmatrix} -\left(1-\omega^2 \dfrac{J_0}{K}\right) & 1 \\ -1 & \left(1-\omega^2 \dfrac{J_1}{K}\right) \end{bmatrix} \begin{Bmatrix} \varphi_0 \\ \varphi_1 \end{Bmatrix} = \{0\}. \tag{13}$$

Natural frequencies ω are determined from the condition

$$\mathrm{Det}(\omega) = \frac{\omega^2}{K}\left(J_0 + J_1 - \omega^2 \frac{J_0 J_1}{K}\right) = 0\,. \tag{14}$$

Value ω=0 is related to the rigid body motion. The second eigenvalue of (14) can be presented in an ordinary form

$$\omega = \sqrt{\frac{K}{J}}\,, \tag{15}$$

where according to Eqs. (5) and (14)

$$K = \frac{K_0 K_1}{K_0 + K_1}\,,\ J = \frac{J_0 J_1}{J_0 + J_1} \tag{16}$$

is the equivalent torsional stiffness and the equivalent polar moment of inertia of the condensed model of shaft line, respectively, Fig. 1.

From the first or the second homogenous equation in expression (13) one finds the ratio of the twist angles

$$\frac{\varphi_0}{\varphi_1} = -\frac{J_1}{J_0} \tag{17}$$

that determines the mode shape as a straight line, Fig. 3. The vibration node is specified by distances l_0 and l_1 from the shaft ends. Since $l_0 + l_1 = l$ and

$$\frac{l_0}{l_1} = \left|\frac{\varphi_0}{\varphi_1}\right| \tag{18}$$

yields

$$l_0 = \frac{J_1}{J_0 + J_1} l\,, \;\; l_1 = \frac{J_0}{J_0 + J_1} l\,. \tag{19}$$

2.3. Forced vibrations

The solution of the complex matrix equation (11) for forced vibrations reads

$$\tilde{A}_0 = -\frac{1}{\mathrm{Det}(\Omega)} \frac{M_E}{K}\,, \;\; \tilde{A}_1 = -\frac{1 + i\Omega \frac{c_0}{K} - \Omega^2 \frac{J_0}{K}}{\mathrm{Det}(\Omega)} \frac{M_E}{K}\,, \tag{20}$$

where

$$\begin{aligned} &\mathrm{Det}(\Omega) = D_R + iD_I\,, \\ &D_R = \frac{\Omega^2}{K}\left(J_0 + J_1 - \Omega^2 \frac{J_0 J_1}{K} + \frac{c_0 c_1}{K} \right), \\ &D_I = \frac{\Omega}{K}\left[\frac{\Omega^2}{K}\left(c_0 J_1 + c_1 J_0\right) - \left(c_0 + c_1\right) \right]. \end{aligned} \tag{21}$$

From Eqs. (15) and (16) one finds

$$J_0 + J_1 = \omega^2 \frac{J_0 J_1}{K}\,, \;\; K = \omega^2 \frac{J_0 + J_1}{J_0 J_1}\,. \tag{22}$$

Furthermore, it is convenient to express the damping coefficients in the form

$$c_0 = 2\gamma_0\, J_0 \omega, \;\; c_1 = 2\gamma_1\, J_1 \omega\,, \tag{23}$$

where γ_0 and γ_1 are dimensionless damping coefficients. Substituting Eqs. (23) and the first and second equation of Eqs. (22) into the second and the third equation of Eqs. (21), respectively, yields

$$\begin{aligned} &D_R = \frac{\Omega^2 J_0}{K}\left\{\left[1 - \left(\frac{\Omega}{\omega}\right)^2\right] + 4\gamma_0\gamma_1\right\} \frac{J_0 + J_1}{J_0}\,, \\ &D_I = \frac{\Omega^2 J_0}{K} 2\gamma_e \frac{\omega}{\Omega}\left[\frac{\gamma_0 + \gamma_1}{\gamma_e}\left(\frac{\Omega}{\omega}\right)^2 - 1\right] \frac{J_0 + J_1}{J_0}\,, \end{aligned} \tag{24}$$

where

$$\gamma_e = \frac{\gamma_0 J_0 + \gamma_1 J_1}{J_0 + J_1}. \tag{25}$$

is the equivalent dimensionless damping coefficient.

According to the definition (4) one obtains for the complex cross-sectional torsional moment

$$\tilde{M}_t = K(\tilde{A}_1 - \tilde{A}_0) = \frac{1 - i2\gamma_0 \frac{\omega}{\Omega}}{C + iD} M_E, \tag{26}$$

where

$$\begin{aligned} C &= \left\{ \left[1 - \left(\frac{\Omega}{\omega} \right)^2 \right] + 4\gamma_0\gamma_1 \right\} \frac{J_0 + J_1}{J_0}, \\ D &= 2\gamma_e \frac{\omega}{\Omega} \left[\frac{\gamma_0 + \gamma_1}{\gamma_e} \left(\frac{\Omega}{\omega} \right)^2 - 1 \right] \frac{J_0 + J_1}{J_0}. \end{aligned} \tag{27}$$

In order to rationalize the fraction in Eq. (26), both the nominator and denominator are multiplied with $C - iD$. This leads to

$$\tilde{M}_t = \frac{E - iF}{C^2 + D^2} M_E, \tag{28}$$

where

$$E = C - 2\gamma_0 \frac{\omega}{\Omega} D, \; F = 2\gamma_0 \frac{\omega}{\Omega} C + D. \tag{29}$$

The cross-sectional torsional moment is a time varying function as a result of twist angles (9), i.e.

$$\tilde{M}_t(t) = \tilde{M}_t e^{i\Omega t}. \tag{30}$$

Actual cross-sectional moment is the real part of (30), like the excitation torque (8), i.e.

$$M_t(t) = \frac{M_E}{C^2 + D^2} (E \cos \Omega t + F \sin \Omega t). \tag{31}$$

Expression (31) can be transformed into the amplitude form

$$M_t(t) = \frac{\sqrt{E^2+F^2}}{C^2+D^2} M_E \cos(\Omega t - \varepsilon)\,, \tag{32}$$

where $\varepsilon = \operatorname{arctg}\left(F/E\right)$ is the phase angle. Substituting (29) and (27) into (32) after some manipulations one obtains the relation between the amplitudes of engine excitation torque and shaft response moment

$$M_t = \alpha M_E\,, \tag{33}$$

where

$$\alpha = \frac{\sqrt{1+4\gamma_0^2\left(\dfrac{\omega}{\Omega}\right)^2}}{\left\{\left[1-\left(\dfrac{\Omega}{\omega}\right)^2\right]^2 + 4\gamma_e^2\left(\dfrac{\omega}{\Omega}\right)^2\left[\dfrac{\gamma_0+\gamma_1}{\gamma_e}\left(\dfrac{\Omega}{\omega}\right)^2 - 1\right]^2\right\}^{\frac{1}{2}} \dfrac{J_0+J_1}{J_0}} \tag{34}$$

is the torque transfer factor from the excitation to the response. The damping term $4\gamma_0\gamma_1$ from the first of Eqs. (27) is ignored in (34) as a small quantity of higher order in comparison to the other damping terms.

In case of resonance $\Omega = \omega$ and Eq. (34) is reduced to

$$\alpha_{res} = \frac{\sqrt{1+4\gamma_0^2}}{2\gamma_e\left(\dfrac{\gamma_0+\gamma_1}{\gamma_e} - 1\right)\dfrac{J_0+J_1}{J_0}}\,. \tag{35}$$

In the case of static engine torque $M_E\left(t\right) = M_E$, $\Omega = 0$ and employing (25) one obtains from (34)

$$\alpha_0 = \frac{\gamma_0}{\gamma_e}\frac{J_0}{J_0+J_1} = \frac{\gamma_0 J_0}{\gamma_0 J_0 + \gamma_1 J_1}\,. \tag{36}$$

Additionally, if $\gamma_0 = \gamma_1 = 0$ Eq. (34) is reduced to

$$\alpha_0 = \frac{J_0}{J_0+J_1}\,. \tag{37}$$

The physical meaning of expressions (36) and (37) is explained in Appendix A.

2.4. Simplified transfer factor of engine torque

A reliable determination of the propeller and the engine damping coefficients is a rather difficult task. Therefore, from a practical point of view, it is more convenient to operate with a unique damping coefficient of the propulsion system. It is an empirical approach and the value of the coefficient is estimated using the data acquired by measurements.

Let us now consider the influence of the above simplification on the accuracy in the case of a realistic example with $J_1 = J_0$. The transfer factor in resonance, Eq. (35), is given by $\alpha_{res} = \dfrac{\sqrt{1+4\gamma_0^2}}{2(\gamma_0+\gamma_1)}$. The following numerical examples:

$\gamma_0 = 0.08,\ \gamma_1 = 0.02:\ \alpha_{res} = 5.064,$

$\gamma_0 = 0.09,\ \gamma_1 = 0.01:\ \alpha_{res} = 5.080,$

$\gamma_0 = 0.1,\ \gamma_1 = 0:\ \alpha_{res} = 5.099,$

demonstrate that the influence of the engine damping coefficient to the accuracy of the method is negligible.

Since the propeller damping is much larger than the engine damping, only the propeller damping can be taken into account with a somewhat increased value. By setting accordingly $\gamma_0 = \gamma$ and $\gamma_1 = 0$, Eq. (25) takes the form

$$\alpha = \frac{\sqrt{1+4\gamma^2\left(\dfrac{\omega}{\Omega}\right)^2}}{\left\{\left[1-\left(\dfrac{\Omega}{\omega}\right)^2\right]^2\left(\dfrac{J_0+J_1}{J_0}\right)^2+4\gamma^2\left(\dfrac{\omega}{\Omega}\right)^2\left[\left(\dfrac{\Omega}{\omega}\right)^2\dfrac{J_0+J_1}{J_0}-1\right]^2\right\}^{\frac{1}{2}}} \tag{38}$$

In resonance $\Omega = \omega$ and Eq. (38) is reduced to

$$\alpha_{res} = \frac{\sqrt{1+(2\gamma)^2}}{2\gamma\dfrac{J_1}{J_0}}. \tag{39}$$

For illustration, diagrams of the transfer factor α, Eq. (38), for the case $J_0 = J_1$ and different values of damping coefficient γ are shown in Fig. 4. The values of α for any value of $\gamma > 0$ converge to unity if Ω/ω approaches zero. On the contrary, values of α for $\gamma = 0$ converge to 0.5. The reasons for this are explained in Appendix A. In the former case the shaft rotates uniformly with a constant angular speed and the damping is changed into resistance, while in the latter case the rotation is uniformly accelerated. The above two cases correspond to the propeller in water and propeller in air, Fig. 5. For these reasons, when performing engine tests on the test bed, a special water brake must be used in order to emulate the propeller resistance.

In fact, if the damping coefficient γ approaches zero in vicinity of $\Omega/\omega \approx 0$, the transfer factor α approaches the vertical asymptote $\Omega/\omega = 0$ in the range between 0.5 and 1, as shown in Fig. 4. When γ takes zero value uniform rotation suddenly turns into a uniformly accelerated rotation. This phenomenon occurs in case of fracture of the propeller shaft pin in an outboard engine, for example, due to a collision of the propeller with a solid.

3. Simplified model of propulsion system

3.1. The first torsional natural mode

The shaft line consists of three main shafts, i.e. the propeller shaft, the intermediate shaft and the crankshaft where the corresponding lengths and diameters are shown in Fig. 6. The shaft line model is divided into segments between the propeller, flanges, flywheel and cranks of the crankshaft with lengths l_i. The polar moment of inertia of the shaft cross-section areas are designated by I_i, and the polar moments of inertia of the propeller, flanges, the flywheel and cranks with the corresponding parts of the shaft as lumped masses by J_i.

It is well known that the first torsional mode is dominant for the forced torsional vibration of ship power transmission lines. It can be assumed in the form of a static deformation as a piecewise linear function depending on the shaft torsional stiffness GI_i, Fig. 7. The value of the twist angle ψ_1 is assumed and values of the remained angles $\psi_i, i = 2, 3, ...n$, are determined with respect to ψ_1 depending on the segment length and stiffness. According to Fig. 7 one finds

$$\begin{aligned}
\Delta\psi_2 &= \frac{I_1 l_2}{I_2 l_1}\psi_1, \\
\psi_2 &= \psi_1 + \Delta\psi_2 = \left(1 + \frac{I_1 l_2}{I_2 l_1}\right)\psi_1, \\
\Delta\psi_3 &= \frac{I_2 l_3}{I_3 l_2}\Delta\psi_2 = \frac{I_1 l_3}{I_3 l_1}\psi_1, \\
\psi_3 &= \psi_2 + \Delta\psi_3 = \left(1 + \frac{I_1 l_2}{I_2 l_1} + \frac{I_1 l_3}{I_3 l_1}\right)\psi_1,
\end{aligned} \tag{40}$$

or generally

$$\Delta\psi_i = \frac{I_1 l_i}{I_i l_1}\psi_1,$$
$$\psi_i = \left(1 + \frac{I_1}{l_1}\sum_{j=2}^{i}\frac{l_j}{I_j}\right)\psi_1. \tag{41}$$

The moment of inertia for the assumed mode is not self-equilibrated and therefore an additional moment due to rigid body rotation, ψ_r, has to be included

$$\sum_{i=0}^{n} J_i \ddot{\Psi}_i - \ddot{\Psi}_r \sum_{i=0}^{n} J_i = 0, \tag{42}$$

where $\ddot{\Psi}_i = \psi_i \cos\omega t$ and $\ddot{\Psi}_r = \psi_r \cos\omega t$. From Eq. (42) one obtains

$$\psi_r = \frac{\sum_{i=0}^{n} J_i \psi_i}{\sum_{i=0}^{n} J_i}. \tag{43}$$

Angle ψ_r is drawn in Fig. 2 and the actual twist angles $\varphi_i = \psi_i - \psi_r$ are indicated as well as the vibration node of the first torsional natural mode.

3.2. Natural vibrations

Differential equation of natural torsional vibrations reads

$$GI_x \frac{\partial^2 \phi}{\partial x^2} - J_x \frac{\partial^2 \phi}{\partial t^2} = 0, \tag{44}$$

where G is the shear modulus, I_x is the polar moment of inertia of shaft cross-section and J_x is the mass polar moment of inertia per unit length. Natural vibrations are harmonic, i.e. $\phi = \varphi \sin \omega t$, where φ is the natural mode and ω is the natural frequency. Hence, Eq. (44) is expressed in terms of the amplitude

$$GI_x \frac{\mathrm{d}^2 \varphi}{\mathrm{d} x^2} + \omega^2 J_x \varphi = 0. \tag{45}$$

In order to determine the natural frequency in case of a piecewise linear natural mode, Fig. 7, it is convenient to use Galerkin's method. Accordingly, Eq. (45) is multiplied with φ and integrated along the shaft

$$G\int_0^l I_x \frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}\varphi\,\mathrm{d}x + \omega^2\int_0^l J_x\varphi^2\,\mathrm{d}x = 0\,. \tag{46}$$

Furthermore, partial integration is used for the first term in (46) that leads to

$$G\int_0^l I_x\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2\mathrm{d}x - \omega^2\int_0^l J_x\varphi^2\,\mathrm{d}x = 0\,. \tag{47}$$

The first and the second term in Eq. (47) in fact represent twice the strain energy and kinetic energy, respectively. Thus, one arrives at the Rayleigh's quotient for the determination of the natural frequency

$$\omega^2 = \frac{G\int_0^l I_x\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2\mathrm{d}x}{\int_0^l J_x\varphi^2\mathrm{d}x}\;. \tag{48}$$

In the considered case of the lumped mass model, the integration of terms in (48) can be performed per segments and one can write

$$\int_0^{l_i} I_x\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2\mathrm{d}x = I_i\left(\frac{\Delta\psi_i}{l_i}\right)^2 l_i\;. \tag{49}$$

$$\int_{\frac{1}{2}l_i}^{\frac{1}{2}l_{i+1}} J_x\varphi^2\mathrm{d}x = J_i\varphi_i^2\,. \tag{50}$$

By taking into account (41) Eq. (48) takes the form

$$\omega^2 = G\left(\frac{I_1\psi_1}{l_1}\right)^2\frac{\sum_{i=1}^{n}\frac{l_i}{I_i}}{\sum_{i=0}^{n}J_i\varphi_i^2}\;. \tag{51}$$

3.3. Forced vibrations

The differential equation of motion for forced torsional vibrations reads

$$GI_x\frac{\partial^2\phi}{\partial x^2} - J_x\frac{\partial^2\phi}{\partial t^2} = -\mu(x,t)\,, \tag{52}$$

where $\mu(x,t) = \mu_x \sin \Omega t$ is distributed external torsional moment with amplitude μ_x and forcing frequency Ω. The solution of Eq. (52) is assumed in the same form as the excitation, i.e.

$$\phi(x,t) = f(x) \sin \Omega t \,. \tag{53}$$

Substituting (53) into (52) one obtains

$$GI_x \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \Omega^2 J_x f = -\mu_x \,. \tag{54}$$

The problem of the forced vibrations of the shaft line can be solved by the Rayleigh-Ritz method. For this purpose the total energy equation is formulated

$$E = \frac{1}{2} G \int_0^l I_x \left(\frac{\mathrm{d} f}{\mathrm{d} x} \right)^2 \mathrm{d} x - \frac{1}{2} \Omega^2 \int_0^l J_x f^2 \mathrm{d} x - \int_0^l \mu_x f \,\mathrm{d} x \,, \tag{55}$$

where particular terms represent the strain energy, the kinetic energy and the work of external load, respectively.

The response function can be found by the mode superposition method. Accordingly, it is assumed in the form

$$f = \sum_{i=0}^{\infty} A_i \varphi_i \,, \tag{56}$$

where φ_i are natural modes.

Coefficient A_0 and zero mode $\varphi_0 = 1$ are included to allow for the rigid body motion of the free shaft line. In spite of the fact that usually the excitation frequency is higher than natural frequency of the first elastic mode, $\omega_1 < \Omega$, experience shows that in the response spectrum the first mode is dominant. Therefore, due to practical reasons one can take only the first two terms of series (56) into account

$$f = A + B\varphi \,. \tag{57}$$

The total energy of the vibrating system, Eq.(55), is equal to zero if the response function is exact. Otherwise, the energy has to be minimal. Substituting the approximating function (57) into Eq. (55) and setting $\partial E / \partial A = \partial E / \partial B = 0$ two independent equations are obtained since

$$\int_0^l J_x \varphi \, \mathrm{d}\, x = 0 \tag{58}$$

as a result of orthogonally of the two natural modes φ_j and φ_k of different indices $j \neq k$, Eq. (56). From the first of those equations, it follows that

$$A = -\frac{\int_0^l \mu_x \mathrm{d}\, x}{\Omega^2 \int_0^l J_x \mathrm{d}\, x} \tag{59}$$

and from the second equation, by taking relation (47) into account

$$B = \frac{1}{\omega^2 - \Omega^2} \frac{\int_0^l \mu_x \varphi \, \mathrm{d}\, x}{\int_0^l J_x \varphi^2 \, \mathrm{d}\, x} \, . \tag{60}$$

Since the distributed mass and excitation are modelled as lumped quantities, integrals in Eqs. (59) and (60) of the continuous functions can be expressed by the summation of discrete quantities. Hence, one can write

$$A = -\frac{\sum_{j=1}^{n} M_j}{\Omega^2 \sum_{i=0}^{n} J_i} \tag{61}$$

$$B = \frac{1}{1 - \left(\dfrac{\Omega}{\omega}\right)^2} \frac{\sum_{j=1}^{n} M_j \varphi_j}{\omega^2 \sum_{i=0}^{n} J_i \varphi_i^2} \, , \tag{62}$$

where M_j is the cylinder excitation torque, φ_j is the corresponding twist angle, and n is the total number of cylinders. The numerator of (61) is the total engine torque, M_E, and the numerator of (62) is the work of engine torque on an average twist angle of the crankshaft, $M_E \varphi_E$.

By defining the constants A and B, Eqs. (61) and (62), the twist angle function f, Eq. (57), is determined, and one can write for the cross-sectional moment in the i-th shaft segment according to definition

$$M_{ti} = GI_i\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)_i = GI_iB\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)_i = K_iB(\varphi_i - \varphi_{i-1}), \tag{63}$$

where $K_i = GI_i/l_i$ is the shaft segment stiffness. Substituting Eq. (62) into (63), yields

$$M_{ti} = \alpha_i M_E, \tag{64}$$

where

$$\alpha_i = \frac{1}{1-\left(\frac{\Omega}{\omega}\right)^2}\frac{K_i(\varphi_i - \varphi_{i-1})}{\omega^2\sum_{i=0}^{n} J_i\varphi_i^2} \tag{65}$$

is the modal transfer factor for undamped vibrations.

In order to include damping in Eq. (65), it is necessary to take damping energy into account in Eq. (55) which is a rather complicate task. This problem can be overcome in a simple way by using analogy with the condensed shaft line model. The structure of Eq. (65) is the same as that of Eq. (38) if the damping coefficient $\gamma = 0$. The fraction in (65)

$$\mu_i = \frac{\omega^2\sum_{i=0}^{n} J_i\varphi_i^2}{K_i(\varphi_i - \varphi_{i-1})\varphi_E} \tag{66}$$

corresponds to the fraction of the polar moment of inertia in (38), i.e.

$$\mu_c = \frac{J_0 + J_1}{J_0}. \tag{67}$$

Hence, one can write for the modal transfer factor of the simplified model of shaft line

$$\alpha_i = \frac{\sqrt{1+4\gamma^2\left(\frac{\omega}{\Omega}\right)^2}}{\left\{\left[1-\left(\frac{\Omega}{\omega}\right)^2\right]^2\mu_i^2 + 4\gamma^2\left(\frac{\omega}{\Omega}\right)^2\left[\left(\frac{\Omega}{\omega}\right)^2\mu_i - 1\right]^2\right\}^{\frac{1}{2}}}, \tag{68}$$

where μ_i is given by Eq. (66). In the case of resonance $\Omega = \omega$ and one obtains

$$\alpha_{i\,\mathrm{res}} = \frac{\sqrt{1+(2\gamma)^2}}{2\gamma\left(\mu_i - 1\right)}\,. \tag{69}$$

4. Engine excitation

4.1. Cylinder torque

Engine excitation originates from the pulsating gas combustion in cylinders. The gas pressure is variable and it induces variable gas forces and inertia forces of the crankshaft and connecting rod mechanism. Both forces have radial and tangential components. The tangential component of the cylinder gas force and inertia force causes cylinder torque, as shown in Fig. 8 for one shaft revolution, [12]. The cylinder tangential force can be expanded into Fourier series. The constant part of the tangential force contributes to the ship propulsion, while the variable part causes torsional vibrations of the shaft line.

Instructions for the calculation of amplitudes of cylinder gas and mass forces are given by engine producers for each type of engine. The amplitudes must be summed up vectorially due to the different phase angles of different harmonics. Usually gas forces are successfully approximated by 25 harmonics, while for the approximation of the inertia forces 5 harmonics are sufficient.

The variable part of the tangential force is specified for one cylinder as a specific force reduced to the piston area. It is given in the form of trigonometric series, [1]

$$T(t) = \sum_{v=1} T_v \sin(v\Omega t + \alpha_v)\,, \tag{70}$$

where T_v is amplitude, $v\Omega$ is the forcing frequency and α_v is the phase angle of the v-th harmonic.

The cylinder torque is defined as

$$M(t) = \frac{1}{2} S A_C \sum_{v=1} T_v \sin(v\Omega t + \alpha_v)\,, \tag{71}$$

where A_C is the piston area and S is the stroke.

4.2. Primary engine torque

In the case of the condensed shaft line model it is assumed that the torque of each cylinder acts in the middle of the engine. The total engine torque is obtained as sum of torques of *n* cylinders, which are time shifted according to the firing order

$$M_E(t) = \sum_{k=1}^{n} M_k(t - t_k). \tag{72}$$

Within one period of shaft rotation, $T_v = 2\pi/\Omega$, the time shift of the *k*-th cylinder is

$$t_k = (k-1)\frac{T}{n}, \; k = 1, 2, \ldots n. \tag{73}$$

Substituting Eqs. (71) and (73) into (72) one obtains

$$M_E(t) = \frac{1}{2} S A_C \sum_{k=1}^{n} \sum_{\nu=1} T_\nu \sin(\nu\Omega t + \alpha_\nu - 2\pi(k-1)\frac{\nu}{n}) . \tag{74}$$

Introducing substitutions

$$\alpha = \nu\Omega t + \alpha_\nu, \; \beta = 2\pi(k-1)\frac{\nu}{n} \tag{75}$$

and employing the trigonometric identity

$$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta \tag{76}$$

Eq. (74) is transformed into the form

$$M_E(t) = \frac{1}{2} S A_C \sum_{k=1}^{n} T_\nu c_\nu \sin(\nu\Omega t + \alpha_\nu - \varepsilon_\nu), \tag{77}$$

where

$$
\begin{aligned}
c_\nu &= \sqrt{a_\nu^2 + b_\nu^2}\,,\\
a_\nu &= \sum_{k=1}^{n} \cos(2\pi(k-1)\frac{\nu}{n}),\\
b_\nu &= \sum_{k=1}^{n} \sin(2\pi(k-1)\frac{\nu}{n}),\\
\varepsilon_\nu &= \operatorname{arctg}\frac{b_\nu}{a_\nu}.
\end{aligned} \tag{78}
$$

In such a way the influence of the time shift is transferred from the cylinder time functions, Eq. (74), to the cylinder torque amplitude, Eq. (77). This enables formulation of the engine torque as an algebraic summation of the cylinder torques for one excitation harmonic.

If the ordinary number of an excitation harmonic coincides with the number of cylinders, i.e. if $\nu = n$ then $a_n = n$, $b_n = 0$, Eqs. (78). In this case Eq. (77) can be written in the form

$$
M_E(t) = \frac{1}{2} S A_C n T_n \sin(n\Omega t + \alpha_\nu) + \frac{1}{2} S A_C \sum_{\nu=1} (1-\delta_{\nu n}) T_\nu c_\nu \sin(\nu\Omega t + \alpha_\nu - \varepsilon_\nu)\,, \tag{79}
$$

where $\delta_{\nu n}$ is the Kronecker symbol ($\delta_{\nu n} = 1$ if $\nu = n$, $\delta_{\nu n} = 0$ if $\nu \neq n$). As a result, the extracted first term in (79) is decisive and of the primary importance. According to (79) the amplitude of the engine torque, acting in the middle of the engine, reads

$$
M_E = \frac{1}{2} S A_C n T_n\,. \tag{80}
$$

4.3. Secondary engine torques

In the simplified model of shaft line, the crankshaft is modelled with lumped masses, Fig. 6. The cylinder torques are distributed along the crankshaft as lumped dynamic loads and are time shifted. In order to be more precise, the secondary engine torques, which are generated in case $\nu \neq n$, have to be derived from the very beginning. For this purpose the energy approach is more convenient.

Starting from the differential equation of torsional vibrations (52), one can write for the total work of the external dynamic load within one period T

$$W_E = \int_0^T \int_0^L \mu(x,t)\phi(x,t)\mathrm{d}x\mathrm{d}t \ . \tag{81}$$

In case of cylinder lumped excitation torques, $M_v(t)$, Eq. (81) can be written in the following form for the ν-th harmonic

$$W_E = \int_0^T \sum_{k=1}^{n} M_{Cv} \sin(\nu\Omega(t-t_k))\varphi_{j(k)} \sin(\nu\Omega t)\mathrm{d}t \ , \tag{82}$$

where M_{Cv} is the amplitude of the ν-th cylinder torque equal for all cylinders, t_k is the firing time shift, Eq. (73), and $\varphi_{j(k)}$ is the amplitude of the corresponding twist angle. The temporal index k = 1, 2... n, denotes the firing order number. The spatial index j denotes the position of the cylinder exposed to the k-th firing.

Setting

$$\alpha = \nu\Omega t, \ \beta = \nu\Omega t_k \ . \tag{83}$$

and employing the trigonometric identity (76), yields

$$W_v = M_{Cv} \int_0^T (A_n \sin^2 \alpha - B_n \sin\alpha \cos\alpha)\mathrm{d}t \ , \tag{84}$$

where

$$\begin{aligned} A_n &= \sum_{k=1}^{n} \varphi_{j(k)} \cos\beta, \\ B_n &= \sum_{k=1}^{n} \varphi_{j(k)} \sin\beta. \end{aligned} \tag{85}$$

The first integral of trigonometric functions in (84) is $I_1 = T/2$, while the second one $I_2 = 0$, due to the orthogonality of the integrating functions. Finally, taking into account the substitutions (73), (83) and (85), one arrives at

$$W_v = \frac{\pi}{\Omega} M_{Cv} \varphi_E A_v \ . \tag{86}$$

where

$$A_\nu = \sum_{k=1}^{n} \frac{\varphi_{j(k)}}{\varphi_E} \cos(2\pi(k-1)\frac{\nu}{n}) . \tag{87}$$

On the other hand, the work of the resulting engine torque is

$$W_\nu = \int_0^T M_{E\nu}\varphi_E \sin^2(\nu\Omega t)\mathrm{d}t = \frac{\pi}{\Omega} M_{E\nu}\varphi_E . \tag{88}$$

Equating Eqs. (86) with (88), yields

$$M_{E\nu} = M_{C\nu} \sum_{k=1}^{n} \frac{\varphi_{j(k)}}{\varphi_E} \cos(2\pi(k-1)\frac{\nu}{n}) . \tag{89}$$

5. Illustrative numerical example

5.1. Characteristics of propulsion system

A numerical example for the verification of the presented analytical procedures for torsional vibration analysis of a shaft line is taken from [14]. Power transmission system of a 2600 TEU Container Ship, equipped with a slow-speed two-stroke seven-cylinder main engine, MAN B&W 7S70 MC-C type, and a five-blade propeller, is considered. The engine power is P = 21735 kW and the nominal speed N_0 = 91 rpm. The main characteristics of the propulsion system are presented in Table 1.

The model of the shaft line, which includes the propeller shaft, the intermediate shaft and the crankshaft, is shown in Fig. 9. The values of the main particulars are listed in Table 2, where the polar moment of inertia of shaft cross-section is $I = D^4\pi/32$. The propeller polar moment of inertia is increased by 38% due to the added mass of surrounding water according to [21]. The polar moment of inertia J_i ,i = 0, 1...12, Fig. 12, include the lumped masses with the corresponding parts of the shaft mass.

5.2. Natural vibrations

The first natural frequency of the torsional shaft line vibration determined by the measurements and by different calculation procedures is listed in Table 3, [14]. The measured value is estimated by the resonant rotation speed technique. A detailed calculation of natural frequency is performed using the Finite Element Method (FEM), by an independent design office, [14]. Four approximate values of natural frequency are obtained considering infinite and finite crankshafts, and two equivalent shaft diameter estimations, [14].

The discrepancies between the numerical results compared to the measured ones show that FEM analysis and simplified procedures taking finite crankshaft stiffness into account give underestimated values. In the case of assumed infinite crankshaft stiffness the obtained values are overestimated. Hence, the measured value of the natural frequency is bounded by the estimated values determined by the simplified procedures from [14].

In this paper the value of natural frequency is estimated for the condensed model of the propulsion system, Section 2.2, by employing formula (15). The equivalent stiffness and mass polar moment of inertia are $K = 84.28 \cdot 10^6$ Nm and $J = 8.3 \cdot 10^4$ kgm^2 respectively. (J_0 in (16) includes $J_0 + J_1$ from Fig. 9, and J_1 the remaining of the moments of inertia.) As a result, one obtains ω = 31.865 rad/s = 5.071 Hz that agrees very well with the measured value of 5.03 Hz.

In order to estimate the first natural frequency of torsional vibration of the shaft line by the simplified procedure presented in Section 3, the approximate shape of the first torsional natural mode is determined in Table 4 according to Section 3.1. The natural mode is shown in Fig. 11. The natural frequency is calculated by Eq. (51) employing the shaft line parameters from Table 2. The value of the natural frequency of 5.059 Hz is very close to the measured value of 5.03 Hz, Table 3.

5.3. Primary engine torque

For the determination of the considered engine excitation the available data of one of the engine producers for a similar engine are used. The amplitude of the relative tangential force per piston area in Eq. (77) is given in the polynomial form

$$T_\nu = A + Bp + Cp^2 + Dp^3 \,. \tag{90}$$

The values of the coefficients, A, B, C and D are tabulated for each the ν-th harmonic. Symbol p in (90) designates the mean indicated pressure, which is determined by formula

$$p = p_0 \left[(1-\varepsilon)\left(\frac{N}{N_0}\right)^2 + \varepsilon \right], \; \varepsilon = 0.0243 \,, \tag{91}$$

where p_0 [bar] is given for particular engine at the nominal engine speed N_0. Values of T_v, Eq. (90), are in [N/mm^2].

The values of the constants in Eq. (90) for the primary excitation harmonic $\nu = n = 7$ are the following: $A = 6.192 \cdot 10^{-2}$, $B = 1.486 \cdot 10^{-4}$, $C = 1.193 \cdot 10^{-3}$, $D = -3.524 \cdot 10^{-5}$. The value of p_0 is 20 bar, Table 1. Diagrams of the indicated pressure p, Eq. (91), and the primary engine torque, M_E, Eq. (80), as a function of engine speed, are shown in Fig. 10. The cylinder torque of the 1st order, which includes both gas force and inertia force, is also included in Fig. 10 as a reference for later use.

5.4. Secondary engine torque

Among large number of the secondary engine torques, $\nu \neq n$, the torque of the 4th order is of particular interest according to the measured shaft line vibrations. The values of constants for the relative tangential pressure force, Eq. (90), are the following: $A = 1.888 \cdot 10^{-1}$, $B = 4.578 \cdot 10^{-3}$, $C = 2.532 \cdot 10^{-3}$, $D = -4.717 \cdot 10^{-5}$.

The relative inertia force is determined by the formula

$$T_{vM} = k_v \frac{m_{OS}}{D_C^2} \frac{D_S}{2} N^2, \tag{92}$$

where

$$k_4 = (-1.745\lambda^4 - 3.491\lambda^2)10^{-6}. \tag{93}$$

Using the data from Table 1 for quantities in Eq. (92), the amplitude of the resulting cylinder force T_{Cv}, consisted of T_{vp} and T_{vM}, is determined by neglecting the phase angle between their vectors as a small quantity in the considered case. The cylinder torque, M_{C4}, is determined by expression (80) and shown in Fig. 10.

In the first consideration firing of one by one cylinder is assumed. The calculation of coefficient A_ν, Eq. (87), is performed in Table 5, taking into account that index $j = k$. The values of the relative twist angle, φ_j / φ_E, are determined using the data from Table 4.

In the second consideration, the given firing order is taken into account, Table 1. The values of the relative twist angle are rearranged in Table 6 according to the firing order. The calculation of the coefficient A_v is performed in Table 7. Since the engine torque, M_{Ev}, depends on A_v, it is obvious that the firing order considerably increases the engine torque.

The value of A_ν from Table 7 gives very large engine torque so that the calculated shear stresses in the intermediate shaft are not comparable with the measured values. The reason for this is that the gradient of the twist angle distribution along the crankshaft is about twice the average gradient of the twist angle determined by the FEM analysis, ϑ_{cal} and ϑ_{FEM} (secant slope) in Fig. 11, respectively. This is because in the calculation, the static deformation of the shaft has been assumed for the natural mode for the Rayleigh-Ritz procedure, whereas with the FEM analysis, a curved natural mode is obtained by solving a dynamic eigenvalue problem. Therefore the value of A_ν must be reduced accordingly.

In order to determine the correct value of A_ν, a relatively simple three mass of the shaft line can be used, Appendix B. In such a way the first elastic mode of the shaft line is represented by three linear pieces, as shown in Fig. B2. Based on the analysis given in Appendix B the coefficient A_ν must be reduced by a factor

$$c = \frac{\varphi_{11}^{(3)} - 1}{\varphi_{11}^{(s)} - 1} = 0.5,$$

where $\varphi_{11}^{(s)}$ and $\varphi_{11}^{(3)}$ are twist angles at the crankshaft end determined by using the static mode shape, and by using the trilinear mode shape, respectively. By applying the correction factor the value of the coefficient A_ν in Table 7 reduces from 0.7157 to 0.3579.

The engine torque, M_{E4}, is determined by formula (89). The diagrams of M_{E4} is shown in Fig. 10. It is interesting to point out that the primary engine torque, M_E, is very large since it is obtained by algebraic summing of the cylinder torques, $M_{C\nu}$. However, the secondary engine torque, $M_{E\nu}$, is even smaller than the cylinder torque, $M_{C\nu}$, as a result of vectorial summation. The secondary engine torque depends on the variation of the twist angle along the crankshaft, while the primary engine torque is not influenced by the crankshaft deformation.

5.5. Primary forced vibrations

The torsional moment in the intermediate shaft, $M_t = \alpha M_E$, is calculated for the condensed model of the propulsion system, Fig. 1, according to the analytical procedure presented in Section 2.3. The values of the transfer factor α, presented by Eq. (38), in which $\mu_C = (J_0 + J_1)/J_0 = 2.213$, are determined by taking the damping coefficient $\gamma = 0.055$, based on MAN B&W experience, into account, Fig. 12. The engine excitation torque of the 7$^{\mathrm{th}}$ order,

M_E, is determined by Eq. (80). Shear stresses in the intermediate shaft, $\tau = M_t / W$, where $W = D_2^3 \pi / 16$ is the section modulus, are shown in Fig. 13.

The measured values of the 7th order shear stress in the intermediate shaft at different engine speeds are also presented in Fig. 13. The resonance peaks in both the calculated and the measured case achieve almost the same value of $\tau_{\max} =$ 60 N/mm^2. In the subresonance domain the measured stresses are somewhat underestimated, while in the first part of the overresonance domain they are overestimated.

Shear stresses determined by FEM analysis are also shown in Fig. 13. In the subresonance domain there are large discrepancies in comparison to the measured values. The resonance peak is very large since obviously a small value of damping coefficient is taken into account in order to be on the safe side.

Due to the extremely large variation of the resonance stress curve shown in Fig. 13, it is rather difficult to evaluate visually the calculated results. Therefore, it is more convenient to compare the calculated engine excitation torque with that based on the measured stresses. The conversion of the measured stresses, τ, into the shaft torsional moment, M_t, and further on into the engine excitation torque, $M_E = M_t / \alpha$, is performed in Table 8. The obtained values of the calculated engine torque, M_{Ec}, and the converted torque, M_{Em}, are compared in Fig. 14. The calculated torque, M_{Ec}, is a smooth approximation of the piecewise curve M_{Em}. Their values are almost equal at the resonance engine speed of N_r = 43.11 rpm.

5.6. Secondary forced vibrations

If the simplified model of the propulsion system for the calculation of forced vibrations is used, the transfer factor of the engine torque to the shaft cross-sectional moment, α_i, is given by Eq. (68). Its values depend on parameter μ_i, Eq. (66), which is a function of the modal twist angle φ_E at the position of the engine excitation torque, M_E. The intermediate shaft, with index i = 2, Fig. 9, is considered now, and one obtains by substituting the previously determined values of particular parameters into (66), $\mu_2 = 3.381 / \varphi_E$. The dependence of μ_2 on the position of the excitation of a particular cylinder is shown in Fig. 15. If M_E is imposed in the middle of engine (cylinder C_4), the value of $\bar{\mu}_2$ = 2.434 is somewhat higher than that in the case of the condensed model, μ_C = 2.213. Consequently, the values of the transfer factor α_2 are somewhat lower, Fig. 12.

Shear stresses in the intermediate shaft, $\tau = M_{E4}/W$, are determined by employing M_{E4} values from the diagram shown in Fig. 10. The stress distribution is shown in Fig. 16 and compared with the measured values. In the subresonance domain the calculated values follow the measured ones to some extent, while in the overresonance domain the calculated values are somewhat lower than the measured values. On the other hand, the calculated resonance peak stress is larger than the measured one. By closely inspecting the measured resonance curve it can be seen that the true peak of the response had not been captured due to the coarse frequency resolution. Discrepancies of the FEM results from the measured ones are quite large, especially in the resonance domain due to the small value of damping coefficient used.

In order to increase the accuracy of the results determined by the simiplified model of propulsion system and to better estimate the measured stresses in the out of resonance domain, it would be necessary to appriximate the twist angle in Eq. (57) by taking more natural modes into account. However, in this case the analytical procedure turns into numerical one. Nevertheless, the most important objective of the analyis is to reliably estimate the resonant response.

6. Conclusion

All classification societies require calculation of the propulsion system operating parameters, but they do not provide simple formulae for the estimation of vibration response. An effort to overcome this shortcoming is presented in a recent publication. The shaft line had been modelled as a two d.o.f. system with the propeller mass on one end of the line and the lumped crankshaft mass at the other. The engine excitation torque and its transfer to the intermediate shaft are determined approximately, Appendix C. As a result, large discrepancies between the calculated and measured shear stresses in the intermediate shaft are obtained in the subresonance domain. Also, the applied value of the dimensionless damping coefficient is almost one half of the empirical value given by the engine producers.

The investigation presented in this self-contained paper is motivated by the state-of-the art in this field. Two mathematical models are offered: one as two d.o.f. system with propeller mass and condensed crankshaft mass, and another multi-d.o.f. system with actual lumped masses. A special attention is paid to calculation of cylinder gas and inertia excitation and the formulation of the resulting engine torque by taking the firing order into account. Two cases are distinguished: the primary engine excitation if the ordinary number of the excitation harmonic coincides with the number of engine cylinders, and the secondary engine excitation if this is not the case. In the former case the engine torque is an algebraic sum of the cylinder torques and it takes a large value. However, in the latter case the engine torque is a vectorial sum of the cylinder torques and is usually small but not negligible.

The vibration of the propulsion system modelled by the condensed model are analysed by the analytical solution of the governing differential equations of motion. For this purpose, the torque transfer factor from the engine to the intermediate shaft is derived consistently. It captures both the case for the propeller in water and the propeller in air, and the primary engine excitation.

The problem of the propulsion system vibrations simulated by a multi d.o.f. model is solved by the energy approach employing the mode superposition technique and the Raleigh-Ritz method. A relatively simple formula for the transfer factor is derived which includes the modal stiffness and mass parameters. By this model it is possible to simulate both primary and secondary shaft line vibrations. It is found that the secondary dynamic response is influenced by both firing order and the crankshaft torsional deformation.

The outlined analytical procedures for vibration analysis of propulsion systems could seem to be a bit theoretical and complicated. However, they are educative and the derived formulae for the first torsional natural frequency, engine excitation torque and the transfer factor are quite simple. Hence, the main results can be obtained quickly and without a computer. The correlation analysis of the calculated results with measured ones show a high accuracy of the presented procedures. Moreover, the derived formulae give more light into the physical background of torsional vibrations of propulsion system, which is not the case if a FE software is used.

The above advantages make the presented analytical procedures for the estimation of propulsion system torsional vibrations a very valuable tool for applications in a preliminary ship design as well as for ships in service.

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Appendix A
Shaft response to propulsion engine torque

The propulsion torque is stationay and presented by the first term of the periodic impulsive total engine torque expanded into Fourier series. The governing differential equations of motion (7) take the following form:

$$\begin{aligned} & M_t - c_0\dot{\phi}_0 - J_0\ddot{\phi}_0 = 0, \\ & M_t + c_1\dot{\phi}_1 - J_1\ddot{\phi}_1 = M_E. \end{aligned} \tag{A1}$$

If damping is present the shaft rotation is uniform with constant angular velocity, $\dot{\phi}_0 = \dot{\phi}_1 = \dot{\phi}$, while the angular accelerations are equal to zer0, $\ddot{\phi}_0 = \ddot{\phi}_1 = 0$. Elimination of M_t from Eqs. (A1), yields

$$\dot{\phi} = \frac{1}{c_0 + c_1} M_E. \tag{A2}$$

By substituting (A2) into one of Eqs. (A1), one obtains

$$M_t = \frac{c_0}{c_0 + c_1} M_E = \frac{\gamma_0 J_0}{\gamma_0 J_0 + \gamma_1 J_1} M_E, \tag{A3}$$

where substitutions (33) are used.

If there is no damping, $c_0 = c_1 = 0$, the shaft rotation is uniformly accelerated, $\ddot{\phi}_0 = \ddot{\phi}_1 = \ddot{\phi}$. Following the above procedure, one arrives at the similar formulae to Eqs. (A2) and (A3), i.e.

$$\ddot{\phi} = \frac{1}{J_0 + J_1} M_E, \ \mathrm{M}_t = \frac{J_0}{J_0 + J_1} M_E. \tag{A4}$$

Transfer factors in Eqs. (A3) and (A4) are different due to different types of shaft motion. They can take the same value only in some special cases.

Appendix B
Three mass model of shaft line

By using a three mass model of the shaft line it is possible to capture also the natural frequency of the second elastic torsional mode. For this purpose, the shaft line is modelled by two beam finite elements with torsional properties, as shown in Fig. B1. The matrix equation of the finite element assembly for the natural vibration analysis reads:

$$\left(\begin{bmatrix} K_0 & -K_0 & \\ -K_0 & K_1 & -K_2 \\ & -K_2 & K_2 \end{bmatrix} - \omega^2 \begin{bmatrix} J_0 & & \\ & J_1 & \\ & & J_2 \end{bmatrix}\right)\begin{Bmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \end{Bmatrix} = \{0\}, \tag{B1}$$

where

$$K_0 = G\frac{I_0}{l_0}, K_2 = G\frac{I_2}{l_2}, K_1 = K_0 + K_2 \tag{B2}$$

are the elements of the stiffness matrix. K_0 is the equivalent stiffness of the propeller shaft and the intermediate shaft, Eq. (5). The determinant of the sum of matrices within parenthesis in Eq. (B1) must vanish

$$Det = A\omega^6 - B\omega^4 + C\omega^2 - D = 0 \tag{B3}$$

where

$$A = J_0 J_1 J_2$$

$$B = J_0 J_1 K_2 + J_1 J_2 K_0 + J_0 J_2 K_1$$

$$C = J_0\left(K_1 K_2 - K_2^2\right) + J_1 K_0 K_2 + J_2 (K_0 K_1 - K_0^2)$$

$$D = K_0 K_1 K_2 - K_0 K_2^2 - K_0^2 K_2 . \tag{B4}$$

Taking into account that $K_1 = K_0 + K_2$ the coefficient *D* vanishes, and the remaining parts of the determinant (B3) can be presented as

$$Det = \omega^2\left(A\omega^4 - B\omega^2 + C\right) = 0 . \tag{B5}$$

The first value $\omega_0 = 0$ is related to the shaft line rotation as a rigid body. The biquadratic equation in (B5) is relevant for the determination of the two natural frequencies of the elastic modes

$$\omega^4 - b\omega^2 + c = 0, \tag{B6}$$

where

$$b=\frac{K_0}{J_0}+\frac{K_1+K_2}{J_1}+\frac{K_2}{J_2}$$

$$c=\frac{J_0+J_1+J_2}{J_0J_1J_2}K_0K_2\ . \tag{B7}$$

Finally, one obtains from (B6)

$$\omega_{1,2}=\sqrt{\frac{b}{2}\pm\sqrt{\left(\frac{b}{2}\right)^2-c}}\ . \tag{B8}$$

By choosing the twist angle φ_1 as the referent one, the values of the remaining two nodal angles are determined from the first and third of Eqs. (B1)

$$\varphi_0=\frac{\varphi_1}{1-\omega^2\frac{J_0}{K_0}},\varphi_2=\frac{\varphi_1}{1-\omega^2\frac{J_2}{K_2}}. \tag{B9}$$

The corresponding data for the three mass model of the considered shaft line are the following:

$l_0=14.77$ m, $l_2=13.69$ m,
$I_0=0.01535\ \mathrm{m}^4$, $I_2=0.04482\ \mathrm{m}^4$,
$J_0=151450\ \mathrm{kgm}^2, J_1=98504\ \mathrm{kgm}^2, J_2=85157\ \mathrm{kgm}^2$.

The value of J_0 includes J_0 and J_1 from Table 2. The resulting mass polar moment of inertia of the moments J_i, with $i=2,3,...11$, in Table 2, is split into two nodal values J_1 and J_2 according to the position of the centre of gravity.

By substituting the above data into Eqs. (B2), (B7), and (B8) one obtains the following values of the natural frequencies: $\omega_1=4.867$ Hz, $\omega_2=12.591$ Hz. The first natural frequency is included in Table 3 and it somewhat underestimates the measured value, like the FEM result does. The second natural frequency is much smaller than the FEM value of 19.383 Hz.

The first two elastic torsional natural modes determined by employing Eqs. (B9) are shown in Fig. B2 and compared with the FEM modes. The agreement of the two mode shapes can be considered qualitatively acceptable.

If the second natural frequency is determined by the Rayleigh's quotient, (Eq 48), using the corresponding natural mode obtained by the three mass model, Fig. B2, its value is increased to $\omega_2=17.88$ Hz, which is in a closer agreement with the FEM value.

Appendix C

Comment on the state-of-the-art in estimation of torsional propulsion system vibrations

In the recent reference [14] torsional vibrations of a propulsion system are analysed by two simplified models, i.e. one with an infinite crankshaft stiffness and the other with a finite stiffness of the crankshaft. Both models are two-mass systems similar to the condensed shaft line model shown in Fig. 1. In case of natural vibrations some tolerable differences of natural frequencies are present, Table 2.

However, concerning the forced vibrations, the 1st order cylinder torque (whose frequency is the engine angular speed) is taken into account for simplicity. The value of the cylinder torque is noticeably lower than the engine torque of the 7-th order, as can be seen in the diagrams shown in Fig. 10. This shortage is traditionally compensated by using the displacement magnification factor of the single mass system in order to achieve a better agreement with the measured shear stress values.

In the case of a single d.o.f. system, shown in Fig. C1, one has to distinguish the magnification factor, α_x, of the static displacement, x_{st}, to the amplitude of the dynamic displacement, x, and the transfer factor α_F, of the amplitude of the external excitation force, F_e, to the amplitude of the elastic spring force, F_S, [22]. These factors read

$$\alpha_x = \frac{x}{x_{st}} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + 4\gamma\left(\frac{\omega}{\omega_0}\right)^2}} \tag{C1}$$

$$\alpha_F = \frac{F_S}{F_e} = \frac{\sqrt{1+4\gamma^2\left(\frac{\omega}{\omega_0}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + 4\gamma\left(\frac{\omega}{\omega_0}\right)^2}} \tag{C2}$$

where ω is the forcing frequency and ω_0 is the natural frequency. Diagrams of α_x and α_F are shown in Figs. C2 and C3, respectively.

By comparing the diagram of the magnification factor of the single d.o.f. system α_x, Fig. C2, used in [14] with the transfer factor for the two d.o.f. system, α, presented by Eq. (38)

and shown in Fig. 4, a difference is obvious. In the subresonant domain, the values of α_x are increased as ω increases, whereas the values of α first decreases and then increases with increasing frequency.

Based on the above remarks one can conclude that the results obtained by the mathematical model for the estimation of torsional vibrations of the propulsion system presented in [14] can be used as a first approximation.

Table 1. Characteristics of propulsion system

Engine	Number of cylinders	$n = 7$
	Nominal speed	$N_0 = 91$ rpm
	Mean indicated pressure	$p_0 = 20$ bar
	Stroke	$S = 2800$ mm
	Cylinder bore diameter	$D_C = 700$ mm
	Oscillating mass per cylinder	$m_{OS} = 7972$ kg
	Connecting rod ratio	$\lambda = 0.488$
	Flywheel moment of inertia	$J_F = 14455$kgm^2
	Crank shaft diameter	$D_{CS} = 840$mm
	Firing order	1725436
Propeller	Diameter	$D_p = 7.42$ m
	Number of blades	$z = 5$
	Mass	$m_p = 33700$ kg
	Moment of inertia in air	$J_{pa} = 107200$ kgm^2
	Propeller shaft diameter	$D_{ps} = 675$ mm
	Intermediate shaft diameter	$D_{is} = 595$ mm

Table 2. Main particulars of shaft line

i	Length of shaft segments, l_i [m]	Shaft diameter, D_i [mm]	Polar moment of inertia of shaft cross-section, I_i [m^4]	Polar moment of inertia of lumped masses, J_i [kgm^2]
0	-	-	-	149000
1	7.516	675	0.02038	2450
2	7.254	595	0.01230	15300
3	0.903	839	0.04865	7800
4	1.240	822	0.04482	22800
5	1.609	822	0.04482	22800
6	1.653	822	0.04482	22800
7	1.774	822	0.04482	22800
8	1.774	822	0.04482	22800
9	1.653	822	0.04482	22800
10	1.716	822	0.04482	22800
11	1.370	822	0.04482	962
$\sum$	28.462			335112

Table 3. Comparison of the first natural frequency of the shaft line

No.	Method	ω [Hz]	Discrepancy related to measurement ε [%]
1	Measurement	5.03±0.1	–
2	Detailed FEM analysis	4.858	-3.42
3	Infinite crankshaft stiffness, 1st simplification, Ref. [14]	5.269	+4.75
4	Finite crankshaft stiffness, 2nd simplification, Ref. [14]	5.330	+5.96
5	Finite crankshaft stiffness, 2nd simplification, Ref. [14]	4.798	-4.61
6	Infinite crankshaft stiffness, 1st simplification, Ref. [14]	4.854	-3.50
7	Three mass model, Appendix B	4.867	-3.24
8	Present condensed model	5.071	+0.81
9	Present simplified model	5.059	+0.57

Table 4. Calculation of shaft line parameters

i	$\frac{l_i}{I_i}$	$\sum_{j=2}^{i}\frac{l_j}{I_j}$	ψ_i, Eq. (41)	$J_i\psi_i$	$\varphi_i=\psi_i-\psi_r$	$J_i\varphi_i^2$
0				0	-1.64	401035
1	368.793		1	2450	-0.64	1005
2	589.756	589.756	2.599	39767	0.959	14058
3	18.561	608.317	2.649	20666	1.009	7939
4	27.666	635.983	2.724	62119	1.084	26787
5	35.899	671.883	2.822	64338	1.181	31815
6	36.881	708.763	2.922	66618	1.281	37429
7	39.581	748.344	3.029	69065	1.389	43962
8	39.581	787.925	3.136	71512	1.496	51021
9	36.881	824.805	3.236	73792	1.596	58070
10	38.286	863.092	3.340	76159	1.700	65871
11	30.567	893.659	3.423	3293	1.783	3057
$\sum$	1262.452			549779		742049

Table 5. Determination of coefficient A_ν, Eqs. (87), n = 7, ν = 4, Fig. 11, firing order: one by one

Coordinate i	Cylinder	Coordinate $j = k$	Relative twist angle φ_j/φ_E	Phase angle $\beta = 2\pi(j-1)\frac{\nu}{n}$	$\cos\beta$	$\frac{\varphi_j}{\varphi_E}\cos\beta$
4	C_7	1	0.780	0	1	0.7800
5	C_6	2	0.850	3.590	-0.9010	-0.7658
6	C_5	3	0.922	7.181	0.6235	0.5748
7	C_4	4	1	6.463	-0.2225	-0.2225
8	C_3	5	1.077	14.362	-0.2225	-0.2396
9	C_2	6	1.149	17.952	0.6235	0.7164
10	C_1	7	1.224	21.542	-0.9010	-1.1028
$\sum$					$a_\nu = 0$	A_ν = -0.2595

Table 6. Relative modal twist angle of crankshaft rearranged according to cylinder firing order 1725436, Fig. 11

Coordinate i	4	5	6	7	8	9	10
Cylinder	C_7	C_6	C_5	C_4	C_3	C_2	C_1
Firing order k	2	7	4	5	6	3	1
Coordinate j	1	2	3	4	5	6	7
Relative twist angle, φ_j/φ_E	0.780	0.850	0.922	1.0	1.077	1.149	1.224

Table 7. Determination of coefficient A_ν, Eqs. (87), n = 7, ν = 4, firing order 1725436

k	$\varphi_{j(k)}/\varphi_E$	$\beta = 2\pi(j-1)\frac{\nu}{n}$	$\cos\beta$	$\frac{\varphi_{j(k)}}{\varphi_E}\cos\beta$
1	1.224	0	1	1.2240
2	0.780	3.590	-0.9010	-0.7028
3	1.149	7.181	0.6235	0.7164
4	0.922	10.771	-0.2225	-0.2051
5	1.0	14.362	-0.2225	-0.2225
6	1.077	17.952	0.6235	0.6715
7	0.850	21.542	-0.9010	-0.7658
$\sum$			$a_\nu = 0$	A_ν = 0.7157

Table 8. Conversion of measured stress in intermediate shaft into engine excitation torque, condensed model

Item i	N [rpm]	Measured τ [N/mm^2]	Based on measurement M_t [MNm]	α (γ=0.055)	M_{Em} [MNm]
1	24.7	3.81	0.1576	0.6729	0.2342
2	27.2	4.83	0.1997	0.7510	0.2659
3	30.7	6.37	0.2634	0.9172	0.2871
4	34.5	9.63	0.3983	1.2579	0.3166
5	38.3	16.69	0.6903	2.1180	0.3259
6	39.5	23.93	0.9897	2.7135	0.3647
7	41.5	43.06	1.8033	4.8500	0.3718
8	42.9	53.32	2.2053	6.9370	0.3179
9	43.1	55.37	2.2901	6.9193	0.3350
		(60.00)*	(2.4816)	(7.5)	0.3350
10	43.9	42.36	1.7520	5.7958	0.3023
11	44.4	35.77	1.4794	4.8657	0.3040
12	45.0	29.71	1.2288	3.9149	0.3139
13	46.1	19.76	0.8172	2.7788	0.2941
14	47.1	15.22	0.6295	2.1581	0.2917
15	49.0	11.33	0.4686	1.4866	0.3152
16	50.5	9.77	0.4041	1.1811	0.3421
17	52.2	8.58	0.3548	0.9510	0.3731
18	55.0	6.20	0.2564	0.7190	0.3565
19	57.1	5.84	0.2415	0.5984	0.4036
20	58.2	5.87	0.2428	0.5488	0.4424
21	59.9	5.50	0.2275	0.4851	0.4690
22	62.1	4.77	0.1973	0.4199	0.4698
23	64.2	4.64	0.1919	0.3707	0.5176
24	66.4	4.13	0.1708	0.3290	0.5191
25	68.0	3.98	0.1646	0.3034	0.5424
26	70.6	3.99	0.1650	0.2684	0.6146
27	71.8	4.34	0.1795	0.2545	0.7051
28	74.6	4.18	0.1729	0.2264	0.7636
29	75.8	4.32	0.1786	0.2159	0.8272
30	78.1	4.02	0.1662	0.1979	0.8399
31	80.2	3.85	0.1592	0.1835	0.8675
32	81.7	3.86	0.1596	0.1742	0.9159
33	83.3	3.81	0.1576	0.1652	0.9540
34	86.0	3.77	0.1559	0.1515	1.0286
35	87.5	3.71	0.1534	0.1447	1.0597
36	89.6	3.60	0.1489	0.1360	1.0945
37	91.2	3.47	0.1435	0.1299	1.1043
38	91.7	3.34	0.1381	0.1281	1.0778
39	92.9	3.28	0.1356	0.1239	1.0941
40	93.8	3.27	0.1352	0.1209	1.1179

* Resonant peak of spline through discrete measured values, [14]

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