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Relaxation of Ginzburg–Landau functional perturbed by continuous nonlinear lower-order term in one dimension

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We study the asymptotic behavior as $\varepsilon \to 0$ of the Ginzburg–Landau functional $I_A^\varepsilon(v) = \int_0^1(\varepsilon^2 v''^2(s) + W(v'(s)) + A(s,v,v')v^2(s))ds$, where $A(s,v,v')$ is the nonlinear lower-order term generated by certain Carathéodory function $a : (0,1)^2 \times \mathbf{R}^2 \to \mathbf{R}$. We obtain Γ-convergence for the rescaled functionals I_A^ε as $\varepsilon \to 0$ by using the notion of Young measures on micropatterns, which was introduced in 2001 by Alberti and Müller. We prove that for $\varepsilon \approx 0$ the minimal value of I_A^ε is close to $E_0 \int_0^1 A_\infty^{1/3}(s)ds \cdot \varepsilon^{2/3}$, where $A_\infty(s) := \frac{1}{2}A(s,0,-1) + \frac{1}{2}A(s,0,1)$ and where E_0 depends only on W. Further, we use this example to establish some general conclusions related to the approach of Alberti and Müller.

Keywords: Asymptotic analysis; Young measures; Ginzburg–Landau functional; gamma convergence.

Mathematics Subject Classification 2010: 34E15, 49J45

1. Introduction

We consider the Ginzburg–Landau functional

$$I_A^\varepsilon(v) = \int_0^1 (\varepsilon^2 v''^2(s) + W(v'(s)) + A(s,v,v')v^2(s))ds, \tag{1.1}$$

where $v \in \mathrm{H}^2_{\mathrm{per}}(0,1)$, W is the 2-well potential (a nonnegative continuous function such that $W(\zeta) = 0$ if and only if $\zeta \in \{-1,1\}$), and A is given by $A(s,v,v') := \int_0^1 a(s,\sigma,v(\sigma),v'(\sigma))d\sigma$, where $a : (0,1)^2 \times \mathbf{R}^2 \to \mathbf{R}$, $a = a(s,\sigma,\boldsymbol{\xi})$, $s \in \mathbf{R}$, $\sigma \in \mathbf{R}$, $\boldsymbol{\xi} \in \mathbf{R}^2$, is 1-periodic in s and σ. We deal with the problem of calculation of the rescaled asymptotic energies $\mathcal{E}_{A,\mathrm{per}} := \lim_\varepsilon \min\{\varepsilon^{-2/3} I_A^\varepsilon(v) : v \in \mathrm{H}^2_{\mathrm{per}}(0,1)\}$ and $\mathcal{E}_A := \lim_\varepsilon \min\{\varepsilon^{-2/3} I_A^\varepsilon(v) : v \in \mathrm{H}^2(0,1)\}$. We also describe geometric behavior of minimizers for I_A^ε as $\varepsilon \to 0$. Functional (1.1) can be regarded as a nonlinear variant of the functional

$$I_{a_0}^{\varepsilon}(v) = \int_0^1 (\varepsilon^2 v''^2(s) + W(v'(s)) + a_0(s)v^2(s))ds \tag{1.2}$$

considered in [1]. A similar functional in dimension $N \geq 1$,

$$J_{\varepsilon,\sigma}(u) = \int_\Omega (\varepsilon^2|\nabla u(s)|^2 + W(u(s)) + \sigma|(-\Delta)^{-1/2}(u(s) - m)|^2)ds, \tag{1.3}$$

was introduced by Ohta and Kawasaki in [14] in order to model microphase separation of diblock copolymer melts (cf. [4–7, 18]). As discussed in [18], $u : \Omega \to \mathbf{R}$ represents the mass density parameter describing the system of two different covalently joined monomers which make a linear chain — the copolymer molecule (whereby $u(s) = 1$ (respectively, $u(s) = -1$) corresponds to the concentration of the first (respectively, the second) monomer at a point s in a bounded open set $\Omega \subseteq \mathbf{R}^N$). The parameters ε and σ are related to the physical properties of the melt (see [7] for details): ε is proportional to the thickness of the transition regions between the two monomers, while σ is inversely proportional to the square of the number of monomers per molecule. The phenomenon of interest here is the formation of regular patterns (for instance, lamellars or circular tubes) which develop as a result of microphase separation when $\varepsilon \approx 0$ and $0 < \varepsilon \ll \sigma \ll 1$. It is easy to see that, in dimension $N = 1$, (1.3) becomes (1.2), provided $\Omega = (0,1)$, $u = v'$, $\sigma = a_0$ and $m = 0$ (a simplified version of (1.2) was independently introduced by Müller in [12] in the context of coherent solid–solid phase transitions, where it is assumed that a_0 is constant). Thus, as $\varepsilon \to 0$, (1.2) accounts for the energy stored by a one-dimensional physical system occupying the interval $(0,1)$. On the other hand, (1.2) is a typical example of a functional where competition of nonconvexity (which favors oscillations in minimizing sequences) and regularization of higher order occurs. To study the asymptotic behavior of (1.1) as $\varepsilon \to 0$, we apply the method of relaxation over the space of Young measures on micropatterns introduced by Alberti and Müller in [1]. The analysis in [1] shows that, under assumption $a_0 \in \mathrm{L}^1(0,1)$, $a_0(s) \geq \alpha_0 > 0$ for a.e. $s \in (0,1)$, the minimizers v_ε of (1.2) for sufficiently small ε resemble a particular sawtooth function, and satisfy $I_{a_0}^{\varepsilon}(v_\varepsilon) \approx E_0 \int_0^1 a_0^{1/3}(s)ds\varepsilon^{2/3}$, where $E_0 := C_0^{2/3}\mathcal{A}_0^{2/3}$, $C_0 := 3/4$, $\mathcal{A}_0 := 2\int_{-1}^1 \sqrt{W(\zeta)}d\zeta$. For results involving different types of lower-order terms see [15–17].

The purpose of the present paper is two-fold. Our first goal is to show that the program in [1] can be successfully applied to the functional I_A^ε with nonlinear lower-order term A under assumption of continuity of a (cf. Sec. 4). Our second goal is to obtain some further deductions based on the study of the functional I_A^ε, which are of wider interest within the framework of Alberti and Müller (cf. Sec. 6). More precisely, while the approach in [1] relies on the notion of f-uniform approximability (cf. Definition 3.4), in this paper we work with a weaker property (namely, partial f-uniform approximability; cf. Definition 6.3), and we establish its connection to the integral representation of the Γ-limit of $(\varepsilon^{-2/3}I_A^\varepsilon)$ (cf. Theorem 6.9). We find that in some cases it is possible to complete all points of the analysis in [1], whereby partial

f-uniform approximability is recovered (cf. Corollary 6.10 and subsequent remarks), but f-uniform approximability is not used as such. In many places throughout the paper we refer to various results in [1].

2. Some Preliminaries

In this paper measurability always means Borel measurability. We consider a compact metric space (K, d) (the space of patterns), which is the set of all measurable mappings $x : \mathbf{R} \to [-\infty, +\infty]$ (modulo equivalence λ-almost everywhere, where λ is one-dimensional Lebesgue measure), endowed with the metric d defined by

$$d(x_1, x_2) := \sum_{k=1}^{\infty} \frac{1}{2^k \alpha_k} \left| \int_{\mathbf{R}} y_k \left(\frac{2}{\pi} \arctan x_1 - \frac{2}{\pi} \arctan x_2 \right) d\lambda \right|, \qquad (2.1)$$

where (y_k) is a sequence of bounded functions which are dense in $\mathrm{L}^1(\mathbf{R})$, such that the support of y_k is a subset of $(-k, k)$, with $\alpha_k := \|y_k\|_{\mathrm{L}^1} + \|y_k\|_{\mathrm{L}^\infty}$. As shown in [1, p. 806], $\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R})$ continuously imbeds in K for every $p \in [1, +\infty]$. The Banach space $\mathrm{C}(K)$ (respectively, $\mathrm{C}_0(\mathbf{R}^r)$) is the space of all continuous real functions on K (respectively, the space of all continuous real functions on $\mathbf{R}^r$ which vanish at infinity), whose dual is identified with the space of all real Radon measures on K (respectively, all real bounded Radon measures on $\mathbf{R}^r$), denoted by $\mathcal{M}(K)$ (respectively, $\mathcal{M}_b(\mathbf{R}^r)$), endowed with the corresponding weak-star topology. Weak-star topology on $\mathcal{M}(K)$ is induced by the norm ϕ defined in [1, p. 799]. By $\mathcal{P}(K)$ (respectively, $\mathcal{P}(\mathbf{R}^r)$) we denote the set of all probability measures in $\mathcal{M}(K)$ (respectively, $\mathcal{M}_b(\mathbf{R}^r)$). If $\mu \in \mathcal{M}(K)$ (respectively, $\mathcal{M}_b(\mathbf{R}^r)$), by $\|\mu\|$ we denote total variation of μ. If $\Omega \subset \mathbf{R}$ is a measurable set such that $\lambda(\Omega) < +\infty$, by $\mathrm{L}^\infty_{w*}(\Omega; \mathcal{M}(K))$ (respectively, $\mathrm{L}^\infty_{w*}(\Omega; \mathcal{M}_b(\mathbf{R}^r))$) we denote the dual of $\mathrm{L}^1(\Omega; \mathrm{C}(K))$ (respectively, $\mathrm{L}^1(\Omega; \mathrm{C}_0(\mathbf{R}^r))$). The set of all K-valued Young measures (Young measures on micropatterns), denoted by $\mathrm{YM}(\Omega; K)$, is the set of all $\boldsymbol{\nu} \in \mathrm{L}^\infty_{w*}(\Omega; \mathcal{M}(K))$ such that $\nu_s \in \mathcal{P}(K)$ for almost every $s \in \Omega$, where $\boldsymbol{\nu}(s) := \nu_s$, $s \in \Omega$. We always endow it with the weak-star topology of $\mathrm{L}^\infty_{w*}(\Omega; \mathcal{M}(K))$. The basic result about Young measures, known as the fundamental theorem of Young measures, can be found in [2]. The weak-star topology on bounded sets in $\mathrm{L}^\infty_{w*}(\Omega; \mathcal{M}(K))$ is induced by the norm Φ defined in [1, p. 769], and therefore $\mathrm{YM}(\Omega; K)$ is metrized by Φ. The elementary Young measure associated to a measurable map $u : \Omega \to K$ (respectively, $u : \Omega \to \mathbf{R}^r$) is the map $\underline{\delta}_u : \Omega \to \mathcal{M}(K)$ (respectively, $\underline{\delta}_u : \Omega \to \mathcal{M}_b(\mathbf{R}^r)$) given by $\underline{\delta}_u(s) := \delta_{u(s)}$, $s \in \Omega$. We say that a sequence of measurable maps $u^k : \Omega \to K$ generates the Young measure $\boldsymbol{\nu}$, if the sequence of elementary Young measures $(\underline{\delta}_{u^k})$ converges to $\boldsymbol{\nu}$ in the topology of $\mathrm{L}^\infty_{w*}(\Omega; \mathcal{M}(K))$. We say that $\mu \in \mathcal{M}(K)$ is invariant with respect to translations if for every $\tau \in \mathbf{R}$ there holds $T^\# \mu = \mu$, where $\langle T^\#_\tau \mu, g \rangle := \langle \mu, g \circ T_\tau \rangle$, and where $T_\tau : K \to K$ is defined by $T_\tau x(t) := x(t - \tau)$, for $x \in K$ and $t \in \mathbf{R}$. $\mathcal{I}(K)$ denotes the class of all invariant measures in $\mathcal{P}(K)$. If $x \in K$ is periodic, the notation ϵ_x stands for the unique invariant probability measure supported on the orbit of x (which is referred to as to an elementary invariant measure), while $\mathcal{EI}(K)$ stands for the set of all elementary invariant measures in $\mathcal{P}(K)$. By $\mathrm{L}^1_{\mathrm{per}}(0,1)$ (respectively, $\mathrm{H}^2_{\mathrm{per}}(0,1)$) we denote the set of all real functions on $(0,1)$, extended to $\mathbf{R}$ by periodicity, which belongs to $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R})$ (respectively, $\mathrm{H}^2_{\mathrm{loc}}(\mathbf{R})$). Sx' denotes the set of discontinuities of $x' \in K$, and $|Sx'|$ denotes cardinality of the set Sx'. If $r_1, r_2 \in \mathbf{R}$ and $r_1 < r_2$, $\mathcal{S}(r_1, r_2)$ stands for the set of all "sawtooth" functions, i.e. the set of all continuous piecewise affine functions $x : (r_1, r_2) \to \mathbf{R}$ with slope equal to either -1 or 1 at almost every point of the interval (r_1, r_2). By $\mathcal{S}_{\mathrm{per}}(r_1, r_2)$ (respectively, $\mathcal{S}_{\mathrm{per},0}(r_1, r_2)$) we denote the set of all functions in $\mathcal{S}(r_1, r_2)$ with property $x(r_1) = x(r_2)$ (respectively, $x(r_1) = x(r_2) = 0$), extended to $\mathbf{R}$ by periodicity. $\mathrm{Lip}(v)$ stands for the Lipschitz constant of a function $v : \mathbf{R} \to \mathbf{R}$. Finally, we recall the notion of Γ-convergence. If X is a metric space then a sequence of functions $F^\varepsilon : X \to [0, +\infty]$ is said to Γ-converge to F on X (which we write as $F^\varepsilon \xrightarrow{\Gamma} F$) if the following two properties are fulfilled: For every $x \in X$ and a sequence (x^ε) in X such that $x^\varepsilon \to x$ in X there holds $\liminf_\varepsilon F^\varepsilon(x^\varepsilon) \geq F(x)$ (the lower bound); for every $y \in X$ there exists a sequence (y^ε) in X such that $y^\varepsilon \to y$ in X and $\limsup_\varepsilon F^\varepsilon(y^\varepsilon) \leq F(y)$ (the upper bound). If there holds $\limsup_\varepsilon F^\varepsilon(x_\varepsilon) < +\infty$, we say that (x_ε) is a finite-energy sequence (or FE sequence) for (F^ε). Detailed and systematic treatment of this type of convergence can be found in [8].

3. Formulation of the Problem and Plan of the Paper

The main steps in asymptotic analysis of the functional (1.2) can be summarized as follows (cf. [1, p. 779]).

- In Step 1 we characterize the class of all Young measures $\boldsymbol{\nu} \in \mathrm{YM}((0,1); K)$ which are generated by sequences of ε-blowups $s \mapsto R^\varepsilon_s v^\varepsilon$ as $\varepsilon \to 0$, where $R^\varepsilon_s v(t) := \varepsilon^{-1/3} v(s + \varepsilon^{1/3} t)$, $t \in \mathbf{R}$ and $v^\varepsilon \in \mathrm{H}^2_{\mathrm{loc}}(\mathbf{R})$.
- In Step 2 we rewrite the rescaled functionals $\varepsilon^{-2/3} I^\varepsilon_{a_0}(v)$ as $\int_0^1 f^\varepsilon_s(R^\varepsilon_s v) ds$ for a suitable choice of $R^\varepsilon_s v$ and f^ε_s.
- In Step 3 we are to identify the Γ-limit f_s of the sequence (f^ε_s) as $\varepsilon \to 0$ on K for almost every $s \in (0,1)$.
- In Step 4 we are required to determine the Γ-limit F_{a_0} of the sequence $(F^\varepsilon_{a_0})$, where $F^\varepsilon_{a_0} : \mathrm{YM}((0,1); K) \to [0, +\infty]$ defined by $F^\varepsilon_{a_0}(\boldsymbol{\nu}) := \int_0^1 \langle \nu_s, f^\varepsilon_s \rangle ds$, if $\boldsymbol{\nu} = \underline{\delta}_{R^\varepsilon v}$ for some $v \in \mathrm{H}^2_{\mathrm{per}}(0,1)$ ($F^\varepsilon_{a_0}(\boldsymbol{\nu}) := +\infty$, otherwise).
- Finally, in Step 5, we are to find the minimizer for F_{a_0} and prove its uniqueness.

According to [1, Proposition 3.6], $f_s : K \to [0, +\infty]$ is defined by $f_s(x) := \frac{\mathcal{A}_0}{2r} |Sx' \cap (-r, r)| + a_0(s) ⨍_{-r}^{r} x^2(\tau) d\tau$, if $x \in \mathcal{S}(-r, r)$ ($f_s(x) := +\infty$, otherwise). By [1, Proposition 3.1], it is natural to define $F_{a_0} : \mathrm{YM}((0,1); K) \to [0, +\infty]$ by $F_{a_0}(\boldsymbol{\nu}) := \int_0^1 \langle \nu_s, f_s \rangle ds$, if $\nu_s \in \mathcal{I}(K)$ for a.e. $s \in (0,1)$ ($F_{a_0}(\boldsymbol{\nu}) := +\infty$, otherwise). To justify the choice of $F^\varepsilon_{a_0}$ in the Step 4, we rely on the following remarks concerning integral functionals on measurable maps from $(0,1)$ to K (denoted by $\mathrm{Meas}((0,1); K)$) and their extensions to $\mathrm{YM}((0,1); K)$. If $f : (0,1) \times K \to$

$[0, +\infty]$ is Borel measurable, then it defines a functional on $\mathrm{Meas}((0,1); K)$ by $u \mapsto \int_0^1 f(s, u(s))ds$. Then there are two ways to extend such a mapping on $\mathrm{YM}((0,1); K)$: by linearity and by $+\infty$. We quote a variant of Theorem 2.12. In [1] which discusses the lower-semicontinuity and Γ-convergence of both types of extensions.

Theorem 3.1. *Let $f_s : K \to [0, +\infty]$ be given by $f_s(u) := f(s, u(s))$, $s \in (0,1)$ for some Borel measurable $f : (0,1) \times \mathbf{R} \to [0, +\infty]$. Let us define $\mathcal{F}_f, \overline{\mathcal{F}}_f$,: $\mathrm{YM}((0,1); K) \to [0, +\infty]$ by $\mathcal{F}_f(\boldsymbol{\nu}) := \int_0^1 \langle \nu_s, f_s\rangle ds$, if $\boldsymbol{\nu} = \underline{\delta}_u$ for some $u \in \mathrm{Meas}((0,1); K)$ ($\mathcal{F}_f(\boldsymbol{\nu}) := +\infty$, otherwise), $\overline{\mathcal{F}}_f(\boldsymbol{\nu}) := \int_0^1 \langle \nu_s, f_s\rangle ds$ for every $\boldsymbol{\nu} \in \mathrm{YM}((0,1); K)$. If the integrands f^ε satisfy $f_s^\varepsilon \xrightarrow{\Gamma} f_s$ as $\varepsilon \to 0$ on K for almost every $s \in (0,1)$, and if the functions Ef^ε defined by $Ef^\varepsilon(s) := \inf_{x\in K} f^\varepsilon(s,x)$ are equi-integrable on $(0,1)$, then $\mathcal{F}_{f^\varepsilon} \xrightarrow{\Gamma} \overline{\mathcal{F}}_f$ and $\overline{\mathcal{F}}_{f^\varepsilon} \xrightarrow{\Gamma} \overline{\mathcal{F}}_f$ as $\varepsilon \to 0$ on $\mathrm{YM}((0,1); K)$. Both $\mathcal{F}_{f^\varepsilon}$ and $\overline{\mathcal{F}}_{f^\varepsilon}$ verify the lower-bound inequality without any assumption on Ef^ε.*

Next, we adjust the language of Theorem 3.1 to our consideration by introducing the following definition.

Definition 3.2. Consider $F_{f^\varepsilon}^\varepsilon : \mathrm{YM}((0,1); K) \to [0, +\infty]$ defined by $F_{f^\varepsilon}^\varepsilon(\boldsymbol{\nu}) := \int_0^1 \langle \nu_s, f_s^\varepsilon\rangle$, if $\nu_s = \delta_{R_s^\varepsilon v}$ for a.e. $s \in (0,1)$ for some $v \in \mathrm{H}^2_{\mathrm{loc}}(\mathbf{R})$ ($F_{f^\varepsilon}^\varepsilon(\boldsymbol{\nu}) := +\infty$, otherwise), where $f_s^\varepsilon \xrightarrow{\Gamma} f_s$ on K as $\varepsilon \to 0$ for a.e. $s \in (0,1)$. We say that the sequence $(F_{f^\varepsilon}^\varepsilon)$ has the commutation property if the Γ-limit of $(F_{f^\varepsilon}^\varepsilon)$ on $\mathrm{YM}((0,1); K)$ as $\varepsilon \to 0$ exists and it is given by $F_f(\boldsymbol{\nu}) := \int_0^1 \langle \nu_s, f_s\rangle ds$ if $\nu_s \in \mathcal{I}(K)$ for a.e. $s \in (0,1)$ ($F_f(\boldsymbol{\nu}) := +\infty$, otherwise).

Further, if $E \subseteq (0,1)$ is a measurable set, we consider $F_{f^\varepsilon;E}^\varepsilon : \mathrm{YM}(E; K) \to [0, +\infty]$ (respectively, $F_{f;E} : \mathrm{YM}(E; K) \to [0, +\infty]$) defined as $F_{f^\varepsilon}^\varepsilon$ (respectively, F_f), but with $\int_0^1$ replaced by $\int_E$, whereby condition $\boldsymbol{\nu} = \underline{\delta}_{R^\varepsilon v}$ for some $v \in \mathrm{H}^2_{\mathrm{loc}}(\mathbf{R})$ (respectively, $\nu_s \in \mathcal{I}(K)$ for a.e. $s \in (0,1)$) is replaced by condition $\nu_s = \delta_{R_s^\varepsilon v}$ for some $v \in \mathrm{H}^2_{\mathrm{loc}}(\mathbf{R})$ and a.e. $s \in E$ (respectively, $\nu_s \in \mathcal{I}(K)$ for a.e. $s \in E$).

Proposition 3.3 (Locality of Γ-convergence). *If the sequence $(F_{f^\varepsilon}^\varepsilon)$ has the commutation property, then for arbitrary Borel measurable set $E \subseteq (0,1)$ the sequence $(F_{f^\varepsilon;E}^\varepsilon)$ also has the commutation property.*

Proof. The lower bound follows by independence of boundary conditions (cf. [1, p. 813]), Borel regularity of λ and Theorem 3.1. Regarding the proof of the upper bound, we argue as follows. Consider $\boldsymbol{\nu} \in \mathrm{YM}(E; K)$ such that $F_{f;E}(\boldsymbol{\nu}) < +\infty$ and $\boldsymbol{\mu} \in \mathrm{YM}((0,1); K)$ such that $\boldsymbol{\mu}\chi_E = \boldsymbol{\nu}$ and $F_f(\boldsymbol{\mu}) < +\infty$. By assumption there exists a sequence $v^\varepsilon \in \mathrm{H}^2_{\mathrm{loc}}(\mathbf{R})$ such that $\underline{\delta}_{R^\varepsilon v^\varepsilon} \xrightarrow{*} \boldsymbol{\mu}$ in $\mathrm{YM}((0,1); K)$ as $\varepsilon \to 0$ and $\lim_\varepsilon F_{f^\varepsilon}^\varepsilon(\underline{\delta}_{R^\varepsilon v^\varepsilon}) = F_f(\boldsymbol{\mu})$. By [1, Remark 2.5] there holds $\underline{\delta}_{R^\varepsilon v^\varepsilon}\chi_E \xrightarrow{*} \boldsymbol{\nu}$ in $\mathrm{YM}(E; K)$ as $\varepsilon \to 0$. If we define $\mu^\varepsilon(E) := \int_E \langle \delta_{R_s^\varepsilon v^\varepsilon}, f_s^\varepsilon\rangle ds$, then $E \mapsto \mu^\varepsilon(E)$ is a Radon measure. For arbitrary subsequence (not relabeled) of measures (μ^ε)

by construction there holds $\|\mu^\varepsilon\| \le F_f(\boldsymbol{\mu})$ (provided ε is small enough). Therefore there exists a Radon measure μ^0 and a further subsequence (not relabeled) such that $\mu^\varepsilon \xrightarrow{*} \mu^0$ in $\mathcal{M}_b(0,1)$ as $\varepsilon \to 0$, $\|\mu^0\| \le F_f(\boldsymbol{\mu})$, $\lim_\varepsilon \mu^\varepsilon(E) = \mu^0(E)$ for every Borel measurable set $E \subseteq (0,1)$. Then μ^0 and $E \mapsto F_{f;E}(\boldsymbol{\mu})$ are Radon measures such that for every open set $E \subseteq (0,1)$ there holds $\mu^0(E) \le F_{f;E}(\boldsymbol{\mu})$. In effect, the upper bound holds for arbitrary measurable set $E \subseteq (0,1)$. □

As a sufficient condition for the successful completion of the Step 4, Alberti and Müller introduced the following notion (cf. [1, p. 803]).

Definition 3.4. Consider $\psi : K \to [0,+\infty]$. We say that K is ψ-uniformly approximable if for every $\varepsilon > 0$ there exists $h = h(\varepsilon) > 0$ such that for every point $x \in K$ we can find an h-periodic point $\tilde{x} \in K$ which satisfies $⨍_0^h d(T_\tau x, T_\tau \tilde{x}) d\tau \le \varepsilon$ and $⨍_0^h \psi(T_\tau \tilde{x}) d\tau \le ⨍_0^h \psi(T_\tau x) d\tau + \varepsilon$.

The main result in [1] now can be stated as follows.

Theorem 3.5. *K is f_s-uniformly approximable for a.e. $s \in (0,1)$, and there holds $F^\varepsilon_{a_0} \xrightarrow{\Gamma} F_{a_0}$ as $\varepsilon \to 0$ on $\mathrm{YM}((0,1);K)$. Moreover, the conclusion is independent of boundary conditions: in the definition of $F^\varepsilon_{a_0}$ we can replace $\mathrm{H}^2_{\mathrm{per}}(0,1)$ by $\mathrm{H}^2(0,1)$.*

Thus Theorem 3.5 ensures that the sequence $(F^\varepsilon_{a_0})$ has the commutation property. In the next two sections we prove that functionals derived from (1.1) (according to the Step 4 of the approach) also posses such a feature, though our arguments are somewhat different in comparison to those in [1]. In Sec. 4 we obtain Γ-convergence by establishing a kind of asymptotic equivalence of functionals $(\varepsilon^{-2/3} I^\varepsilon_A)$ and suitably chosen (rescaled) functionals of type (1.2). Then, independently of Sec. 4, we identify the Γ-limit in Sec. 5. In the calculations we avoid the question of φ-uniform approximability of K for a natural choice of φ. In the end of our consideration, we revisit the proof of Γ-convergence of functionals (1.1) and we note that the strategy of the proofs in [1] can be followed step by step, even if φ-uniform approximability is not at our disposal. We discuss this topic in some detail in Sec. 6.

4. Γ-Convergence Result

Herein we describe the main points of the approach concerning the functional (1.1). To this end, we consider W which satisfies

$$W(\zeta) \ge c_0|\zeta|^{r_0} \quad \text{for every } \zeta \text{ such that } |\zeta| \ge R_0, \tag{4.1}$$

where $c_0, R_0 > 0$ and $r_0 \ge 1$. We assume that a is a Carathéodory function (measurable in (s,σ), continuous in $\boldsymbol{\xi}$, extended by periodicity to $\mathbf{R}^2 \times \mathbf{R}^2$) such that:

$$\inf_{\boldsymbol{\xi} \in \mathbf{R}^2} a(s,\sigma,\boldsymbol{\xi}) \ge \alpha_0 > 0, \quad \text{for a.e. } (s,\sigma) \in (0,1)^2, \tag{4.2}$$

$$a(s,\sigma,\boldsymbol{\xi}) \le h_0(s,\sigma) + |\xi_1|^q h_1(s) + |\xi_2|^p h_2(s), \quad \text{for a.e. } (s,\sigma) \in (0,1)^2, \tag{4.3}$$

where $\boldsymbol{\xi} = (\xi_1,\xi_2) \in \mathbf{R}^2$, $h_1, h_2 \in \mathrm{L}^1_{\mathrm{per}}(0,1)$, $h_0 \in \mathrm{L}^1_{\mathrm{per}}((0,1)\times(0,1))$, $q \in (0,+\infty)$ and $p \in (0,r_0]$. In this section we prove that $\mathcal{E}_A = \mathcal{E}_{A,\mathrm{per}} =$

$E_0 \int_0^1 (\int_{\mathbf{R}^2} A(s, \boldsymbol{\xi}) d\nu_\infty(\boldsymbol{\xi}))^{1/3} ds$, where $\nu_\infty := \frac{1}{2}\delta_{(0,-1)} + \frac{1}{2}\delta_{(0,1)}$. We begin by proving the following technical results which we use in the proof of Proposition 4.2.

Lemma 4.1. *Consider* $A^\varepsilon_{s,\tau}(x) := \int_0^1 \fint_{-r}^r a(s + \varepsilon^{1/3}\tau, \sigma + \varepsilon^{1/3}t, \varepsilon^{1/3}x(t), x'(t)) dt d\sigma$ *and* $\tilde{A}(s, 0, \cdot) : K \to [0, +\infty]$ *defined by* $\tilde{A}(s, 0, y) := \int_0^1 \fint_{-r}^r a(s, \sigma, 0, y(t)) dt d\sigma$. *If* $x_\varepsilon \to x$ *in* $\mathrm{W}^{1,1}(-r, r)$ *as* $\varepsilon \to 0$, $x \in \mathcal{S}(-r, r)$ *and* $\|x'_\varepsilon\|_{\mathrm{L}^{r_0}(-r,r)} \le C$, *then*

$$A^\varepsilon_{s,\tau}(x_\varepsilon) \rightharpoonup \tilde{A}(s, 0, x') \quad \text{in } \mathrm{L}^1(-r, r) \quad (a.e.\ s \in (0, 1)), \tag{4.4}$$

$$A^{\varepsilon,M}_{s,\tau}(x_\varepsilon) \overset{*}{\rightharpoonup} \tilde{A}^M(s, 0, x') \quad \text{in } \mathrm{L}^\infty(-r, r) \quad (a.e.\ s \in (0, 1)), \tag{4.5}$$

where, for $M > 0$, $\tilde{A}^M(s, 0, y) := \int_0^1 \fint_{-r}^r a^M(s, \sigma, 0, y(t)) dt d\sigma$, $a^M := \min\{a, M\}$ *and* $A^{\varepsilon,M}_{s,\tau}(x) := \int_0^1 \fint_{-r}^r a^M(s + \varepsilon^{1/3}\tau, \sigma + \varepsilon^{1/3}t, \varepsilon^{1/3}x(t), x'(t)) dt d\sigma$.

Proof. We divide the proof into four steps.

Step 1. To begin with, we extract a subsequence (not relabeled) such that $x_\varepsilon(t) \to x(t)$ and $x'_\varepsilon(t) \to x'(t)$ for a.e. $t \in (-r, r)$. By Egoroff's theorem for every $\eta \in (0, 1)$ there exists a measurable set $E_\eta \subseteq (-r, r)$ such that $\lambda(E_\eta) \le \eta$ and $x_\varepsilon \to x$ uniformly on $(-r, r)\backslash E_\eta$ and $x'_\varepsilon \to x'$ uniformly on $(-r, r)\backslash E_\eta$. We fix $M > 0$ and we consider $\tilde{a}^M : (0, 1) \times \mathbf{R}^2 \to \mathbf{R}$ defined by $\tilde{a}^M(s, \boldsymbol{\xi}) := \int_0^1 a^M(s, \sigma, \boldsymbol{\xi}) d\sigma$. By the Scorza–Dragoni theorem there exists an increasing sequence of compact sets (F_j) such that $F_j \subseteq (0, 1)$, $\lim_{j\to 0} \lambda((0, 1)\backslash F_j) = 0$ and such that the restriction of $\tilde{a}^M$ on $F_j \times \mathbf{R}^2$ is continuous. Since $\tilde{a}^M$ is uniformly continuous on compact set $([s - \Delta, s + \Delta] \cap F_j) \times [-2, 2]^2$, for every $\Delta > 0$ there exists $m_0(\Delta, s, j, M) \in \mathbf{N}$ such that for every $m \ge m_0$ and every $(\rho_i, \boldsymbol{\xi}_i) \in ([s - \Delta, s + \Delta] \cap F_j) \times [-2, 2]^2$, $i = 1, 2$, $\|(\rho_1, \boldsymbol{\xi}_1) - (\rho_2, \boldsymbol{\xi}_2)\| \le \frac{1}{m}$ implies $|\tilde{a}^M(\rho_1, \boldsymbol{\xi}_1) - \tilde{a}^M(\rho_2, \boldsymbol{\xi}_2)| \le \Delta$. On the other hand, for a given $m \in \mathbf{N}$ there exists $\varepsilon_0 = \varepsilon_0(m, \eta)$ such that for every $\varepsilon \in (0, \varepsilon_0]$ there holds $\|x_\varepsilon - x\|_{\mathrm{L}^\infty((-r,r)\backslash E_\eta)} \le \frac{1}{2m}$ and $\|x'_\varepsilon - x'\|_{\mathrm{L}^\infty((-r,r)\backslash E_\eta)} \le \frac{1}{2m}$.

Step 2. Further, we prove that for every $\psi \in \mathrm{L}^\infty(-r, r)$ there holds

$$\limsup_{\varepsilon\to 0} \fint_{-r}^r A^\varepsilon_{s,\tau}(x_\varepsilon)\psi(\tau) d\tau \le \fint_{-r}^r \tilde{A}(s, 0, x')\psi(\tau) d\tau. \tag{4.6}$$

Since a is 1-periodic in σ, by Step 1 for arbitrary $M > 0$ we get

$$\limsup_{\varepsilon\to 0} \fint_{-r}^r A^{\varepsilon,M}_{s,\tau}(x_\varepsilon) d\tau \le \frac{1}{2r} \int_{(-r,r)\backslash E_\eta} \tilde{a}^M(s, 0, x'(t)) dt + \Delta_j(M, \eta),$$

where $\Delta_j(M, \eta) := \frac{1}{2r}\Delta + \frac{M}{4r^2}(\lambda(F_j)\lambda(E_\eta) + 2r\lambda((0, 1)\backslash F_j))$. We pass to the limit as $j \to +\infty$ and as $\eta \to 0$ by the dominated convergence theorem, and, by arbitrariness of $\Delta > 0$, we obtain $\limsup_\varepsilon \fint_{-r}^r A^{\varepsilon,M}_{s,\tau}(x_\varepsilon) d\tau \le \tilde{A}(s, 0, x')$. We set $H_0(s, \sigma) := h_0(s, \sigma) + c_1 h_1(s) + c_2 h_2(s)$, where $\|x_\varepsilon\|^q_{\mathrm{L}^q} \le c_1$ and $\|x'_\varepsilon\|^{r_0}_{\mathrm{L}^{r_0}} \le c_2$. Since h_0 is 1-periodic in σ, we have $\int_0^1 H_0(s + \varepsilon^{1/3}\tau, \sigma + \varepsilon^{1/3}t) d\sigma = H(s + \varepsilon^{1/3}\tau)$, where $H(s) :=$

$\int_0^1 H_0(s,\rho)d\rho \in \mathrm{L}^1(0,1)$. To proceed, we estimate

$$\fint_{-r}^{r} A^{\varepsilon}_{s,\tau}(x_\varepsilon)d\tau \leq \fint_{-r}^{r} A^{\varepsilon,M}_{s,\tau}(x_\varepsilon)d\tau + \frac{1}{4r^2}\varepsilon^{-1/3}\int_{s-\varepsilon^{1/3}r}^{s+\varepsilon^{1/3}r} H_M(\theta)d\theta,$$

and it results $\limsup_\varepsilon \fint_{-r}^{r} A^{\varepsilon}_{s,\tau}(x_\varepsilon)d\tau \leq \tilde{A}(s,0,x') + \frac{1}{2r}H_M(s)$, where $H_M(s) := 0$ if $H(s) \leq M$, and $H_M(s) := H(s)$, otherwise. Since $H \in \mathrm{L}^1(0,1)$, there holds $H_M \to 0$ in $\mathrm{L}^1(0,1)$ as $M \to +\infty$, and we can extract a subsequence (not relabeled) such that $H_M(s) \to 0$ for a.e. $s \in (0,1)$. Therefore we can pass to the limit as $M \to +\infty$, getting (4.6) for $\psi(\tau) = 1$. Next, we claim that for every simple function $\psi(\tau) := \sum_{i=1}^{n} c_i \chi_{B_i}(\tau)$, $\tau \in (-r,r)$ (where $B_i \subset (-r,r)$, $i = 1,\ldots,n$, are measurable sets such that $\lambda((-r,r)\backslash \bigcup_{i=1}^{n} B_i) = 0$) (4.6) holds true. By outer Borel regularity of measure λ for every $\delta > 0$ there exists an open set B^i_δ such that $B^i \subseteq B^i_\delta$ and $\lambda(B^i_\delta \backslash B^i) \leq \delta$. Then each B^i_δ can be written as a disjoint union of countably many open intervals I^i_k, $k \in \mathbf{N}$. As before, it follows

$$\int_{I^i_k} A^{\varepsilon}_{s,\tau}(x_\varepsilon)d\tau \leq \int_{I^i_k} A^{\varepsilon,M}_{s,\tau}(x_\varepsilon)d\tau + \frac{1}{2r}\varepsilon^{-1/3}\int_{s-\varepsilon^{1/3}r}^{s+\varepsilon^{1/3}r} H_M(\theta)\chi_{I^i_k}(\varepsilon^{-1/3}(\theta - s))d\theta,$$

where, by [9, Corollary 1.7.2, p. 44], for a.e $s \in (0,1)$ holds

$$\lim_{\varepsilon\to 0} \varepsilon^{-1/3}\int_{s-\varepsilon^{1/3}r}^{s+\varepsilon^{1/3}r} H_M(\theta)\chi_{I^i_k}(\varepsilon^{-1/3}(\theta - s))d\theta = \lambda(I^i_k)H_M(s).$$

As we apply the sum over all k, we have

$$\begin{aligned}\int_{B^i} A^{\varepsilon}_{s,\tau}(x_\varepsilon)d\tau \leq \int_{B^i_\delta} A^{\varepsilon,M}_{s,\tau}(x_\varepsilon)d\tau &+ \sum_{k=1}^{+\infty}\frac{1}{2r}\varepsilon^{-1/3} \\ &\times \int_{s-\varepsilon^{1/3}r}^{s+\varepsilon^{1/3}r} H_M(\theta)\chi_{I^i_k}(\varepsilon^{-1/3}(\theta - s))d\theta.\end{aligned}$$

We pass to the limit as $\varepsilon \to 0$, $\delta \to 0$, and as $M \to +\infty$ by the dominated convergence theorem. Thus, summation over i yields (4.6). For a given $\psi \in \mathrm{L}^\infty(-r,r)$ we take a sequence of simple functions (ψ_N) such that $\|\psi - \psi_N\|_{\mathrm{L}^\infty(-r,r)} \leq \frac{1}{N}$. Then $\fint_{-r}^{r} A^{\varepsilon}_{s,\tau}(x_\varepsilon)\psi(\tau)d\tau \leq \fint_{-r}^{r} A^{\varepsilon}_{s,\tau}(x_\varepsilon)\psi_N(\tau)d\tau + \frac{1}{N}\fint_{-r}^{r} A^{\varepsilon}_{s,\tau}(x_\varepsilon)d\tau$, and, as $\varepsilon \to 0$ we get

$$\limsup_{\varepsilon\to 0}\fint_{-r}^{r} A^{\varepsilon}_{s,\tau}(x_\varepsilon)\psi(\tau)d\tau \leq \fint_{-r}^{r} \tilde{A}(s,0,x')\psi_N(\tau)d\tau + \frac{1}{N}\tilde{A}(s,0,x').$$

As we let $N \to +\infty$, we obtain (4.6).

Step 3. By Step 1 for $\varepsilon \leq \min\{\varepsilon_0, \Delta^3 r^{-3}\}$ $\fint_{-r}^{r} A^{\varepsilon}_{s,\tau}(x_\varepsilon)d\tau$ is bounded from below by $-\frac{\Delta}{2r} + \frac{1}{4r^2\varepsilon^{1/3}}\int_{[s-\varepsilon^{1/3}r, s+\varepsilon^{1/3}r]\cap F_j}\int_{(-r,r)\backslash E_\eta} \tilde{a}^M(\rho,0,x'(t))dtd\rho$. Therefore

$$\liminf_{\varepsilon\to 0}\fint_{-r}^{r} A^{\varepsilon}_{s,\tau}(x_\varepsilon)d\tau \geq -\frac{\Delta}{2r} + \frac{1}{2r}\int_{(-r,r)\backslash E_\eta} \tilde{a}^M(s,0,x'(t))dt\chi_{F_j}(s).$$

By passing to the limit, first as $j \to +\infty$, then as $\eta \to 0$ and as $\Delta \to 0$, and finally as $M \to +\infty$, we get $\liminf_\varepsilon \fint_{-r}^{r} A^{\varepsilon}_{s,\tau}(x_\varepsilon)d\tau \geq \fint_{-r}^{r} \tilde{A}(s,0,x')d\tau$. Now we claim that for every $\psi \in \mathrm{L}^\infty(-r,r)$ there holds

$$\liminf_{\varepsilon\to 0} \fint_{-r}^{r} A^\varepsilon_{s,\tau}(x_\varepsilon)\psi(\tau)d\tau \geq \fint_{-r}^{r} \tilde{A}(s,0,x')\psi(\tau)d\tau. \tag{4.7}$$

By Mikusinski's theorem (cf. [10, Theorem 3.104, p. 113]) for every $\psi \in \mathrm{L}^1(-r,r)$ there exists a sequence of piecewise constant functions (ψ_k) such that $\psi_k \to \psi$ in $\mathrm{L}^1(-r,r)$ as $k \to +\infty$, where $\psi_k(s) = \sum_{i=1}^{N_k} c_i^k \chi_{I_k^i}(s)$, and $I_k^i = (a_k^i, b_k^i)$ for $i = 1,\ldots,N_k$ are pairwise disjoint open intervals. For a subsequence (ψ_{k_M}) such that $\|\psi - \psi_{k_M}\|_{\mathrm{L}^1(-r,r)} \leq \frac{1}{M^2}$ there holds $\fint_{-r}^{r} A^\varepsilon_{s,\tau}(x_\varepsilon)\psi(\tau)d\tau \geq \fint_{-r}^{r} A^{\varepsilon,M}_{s,\tau}(x_\varepsilon)\psi_{k_M}(\tau)d\tau - \frac{1}{M}$. As before, we pass to the limit, first as $\varepsilon \to 0$, and then as $M \to +\infty$, getting (4.7).

Step 4. Since the argument above can be carried out for arbitrary subsequence of the sequence $(A^\varepsilon_{s,\tau}(x_\varepsilon))$, we get (4.4). To prove (4.5) we consider $\psi \in \mathrm{L}^1(-r,r)$ and a sequence (ψ_k) as in Step 3. Then we obtain the following bounds:

$$\fint_{-r}^{r} A^{\varepsilon,M}_{s,\tau}(x_\varepsilon)\psi(\tau)d\tau \geq \fint_{-r}^{r} A^{\varepsilon,M}_{s,\tau}(x_\varepsilon)\psi_k(\tau)d\tau - M\|\psi - \psi_k\|_{\mathrm{L}^1(-r,r)}, \tag{4.8}$$

$$\fint_{-r}^{r} A^{\varepsilon,M}_{s,\tau}(x_\varepsilon)\psi(\tau)d\tau \leq \fint_{-r}^{r} A^{\varepsilon,M}_{s,\tau}(x_\varepsilon)\psi_k(\tau)d\tau + M\|\psi - \psi_k\|_{\mathrm{L}^1(-r,r)}. \tag{4.9}$$

Finally, as we let $\varepsilon \to 0$ and $k \to +\infty$ in (4.8) and (4.9), we recover (4.5). □

Next, we complete the second and the third step of the approach.

Proposition 4.2. *If $v \in \mathrm{H}^2_{\mathrm{per}}(0,1)$, then there holds $\varepsilon^{-2/3} I^\varepsilon_A(v) = \int_0^1 \varphi^\varepsilon_s(R^\varepsilon_s v)ds$, where $\varphi^\varepsilon_s : K \to [0,+\infty]$ is defined by*

$$\varphi^\varepsilon_s(x) := \fint_{-r}^{r} (\varepsilon^{2/3}x''^2(\tau) + \varepsilon^{-2/3}W(x'(\tau)) + A^\varepsilon_{s,\tau}(x)x^2(\tau))d\tau, \tag{4.10}$$

if $x \in \mathrm{H}^2(-r,r)$ ($\varphi^\varepsilon_s(x) := +\infty$, otherwise). Furthermore, under assumptions (4.1)–(4.3), *it follows $\varphi^\varepsilon_s \xrightarrow{\Gamma} \varphi_s$ as $\varepsilon \to 0$ on K for a.e. $s \in (0,1)$, where $\varphi_s : K \to [0,+\infty]$ is defined by*

$$\varphi_s(x) := \frac{\mathcal{A}_0}{2r}|Sx' \cap (-r,r)| + \tilde{A}(s,0,x')\fint_{-r}^{r} x^2(\tau)d\tau, \tag{4.11}$$

if $x \in \mathcal{S}(-r,r)$ ($\varphi_s(x) := +\infty$, otherwise).

Proof. To prove the lower bound, we consider a sequence (x_ε) such that $x_\varepsilon \to x$ in K as $\varepsilon \to 0$. Without loss of generality, we can assume that there holds $\liminf_\varepsilon \varphi^\varepsilon_s(x_\varepsilon) < +\infty$ (otherwise there is nothing to prove). By (4.2) there holds $\liminf_\varepsilon f^\varepsilon_{\alpha_0}(x_\varepsilon) < +\infty$, where $f^\varepsilon_{\alpha_0} : K \to [0,+\infty]$ is defined as in (4.10), with $A^\varepsilon_{s,\tau}(x)$ replaced by α_0. Then there exists a subsequence (not relabeled), such that (x_ε) is FE sequence for $(f^\varepsilon_{\alpha_0})$. By the theorem of Modica and Mortola (cf. [11]) (x_ε) is pre-compact in $\mathrm{W}^{1,1}(-r,r)$, $\liminf_\varepsilon \fint_{-r}^{r}(\varepsilon^{2/3}x''^2_\varepsilon(\tau) + \varepsilon^{-2/3}W(x'_\varepsilon(\tau)))d\tau \geq$

$\frac{\mathcal{A}_0}{2r}|Sx' \cap (-r,r)|$, and $x \in \mathcal{S}(-r,r)$. By Lemma 4.1 the sequence $(A^{\varepsilon,M}_{s,\tau}(x_\varepsilon))$ converges to $\tilde{A}^M(s,0,x')$ weakly-star in $\mathrm{L}^\infty(-r,r)$ as $\varepsilon \to 0$ for a.e. $s \in (0,1)$. Consequently, $x_\varepsilon^2 \to x^2$ in $\mathrm{L}^1(-r,r)$ as $\varepsilon \to 0$ implies that $⨍_{-r}^r A^{\varepsilon,M}_{s,\tau}(x_\varepsilon)x_\varepsilon^2(\tau)d\tau$ converges to $\tilde{A}^M(s,0,x')⨍_{-r}^r x^2(t)dt$ as $\varepsilon \to 0$ for a.e. $s \in (0,1)$. Then it follows $\liminf_\varepsilon ⨍_{-r}^r A^\varepsilon_{s,\tau}(x_\varepsilon)x_\varepsilon^2(\tau)d\tau \geq \tilde{A}(s,0,x')⨍_{-r}^r x^2(t)dt$ for a.e. $s \in (0,1)$ (where we pass to the limit as $M \to +\infty$ by Fatou's lemma). By the uniqueness of the cluster point x we get $\liminf_\varepsilon \varphi_s^\varepsilon(x_\varepsilon) \geq \varphi_s(x)$. The upper bound follows by [1, Proposition 3.6]: for every $\overline{x} \in \mathcal{S}(-r,r)$ there exists a sequence $(\overline{x}_\varepsilon)$ such that $\overline{x}_\varepsilon \in \mathrm{H}^2(-r,r)$, $\mathrm{Lip}(\overline{x}_\varepsilon) \leq 1$, $\overline{x}_\varepsilon \to \overline{x}$ in $\mathrm{W}^{1,1}(-r,r)$ (and therefore $\overline{x}_\varepsilon \to \overline{x}$ in $\mathrm{L}^\infty(-r,r)$) as $\varepsilon \to 0$ and $\lim_\varepsilon f^\varepsilon_{\alpha_0}(\overline{x}_\varepsilon) = f_{\alpha_0}(\overline{x})$, where $f_{\alpha_0} : K \to [0,+\infty]$ is defined as in (4.11), with $\tilde{A}(s,0,x')$ replaced by α_0. By Lemma 4.1 the sequence $(A^\varepsilon_{s,\tau}(\overline{x}_\varepsilon))$ converges to $\tilde{A}(s,0,\overline{x}')$ weakly in $\mathrm{L}^1(-r,r)$ as $\varepsilon \to 0$ for a.e. $s \in (0,1)$, which gives $\lim_\varepsilon ⨍_{-r}^r A^\varepsilon_{s,\tau}(\overline{x}_\varepsilon)\overline{x}_\varepsilon^2(\tau)d\tau = ⨍_{-r}^r \tilde{A}(s,0,\overline{x}')\overline{x}^2(\tau)d\tau$. □

In the next corollary we show that there exist many FE sequences for (φ_s^ε).

Corollary 4.3. *If* (4.1)–(4.3) *hold, and if* $s \mapsto \int_0^1 h_0(s,\sigma)d\sigma \in \mathrm{L}^{p_0}(0,1)$ *and* $h_1, h_2 \in \mathrm{L}^{p_0}(0,1)$ *for some* $p_0 \in (1,+\infty]$, *then for a.e.* $s \in (0,1)$ *there holds*: (x_ε) *is FE sequence for* (φ_s^ε) *if and only if* (x_ε) *is FE sequence for* $(f^\varepsilon_{\alpha_0})$. *If* $r_0 > 1$, *we can allow* $p_0 = 1$.

Proof. The "only if" part follows by (4.2). To prove the "if" part, we consider arbitrary FE sequence (x_ε) for $(f^\varepsilon_{\alpha_0})$. By inequality $\varphi_s^\varepsilon(x_\varepsilon) \leq f^\varepsilon_{\alpha_0}(x_\varepsilon) + ⨍_{-r}^r A^\varepsilon_{s,\tau}(x_\varepsilon)x_\varepsilon^2(\tau)d\tau$ and by (4.3) we estimate $\limsup_\varepsilon \varphi_s^\varepsilon(x_\varepsilon) \leq C + \limsup_\varepsilon ⨍_{-r}^r H_0(s+\varepsilon^{1/3}\tau)x_\varepsilon^2(\tau)d\tau$, where $H_0(s) := \int_0^1 h_0(s,\sigma)d\sigma + c_1h_1(s) + c_2h_2(s)$, $\frac{1}{2r}\|x_\varepsilon\|^q_{\mathrm{L}^q(-r,r)} \leq c_1$, $\frac{1}{2r}\|x'_\varepsilon\|^p_{\mathrm{L}^p(-r,r)} \leq c_2$, and s is the Lebesgue point of $H_0^{p_0}$. Then, by an application of the Rellich imbedding theorem we have $\|x_\varepsilon^2\|_{\mathrm{L}^{q_0}(-r,r)} \leq C_0$, where $\frac{1}{p_0} + \frac{1}{q_0} = 1$. Hence, Hölder's Inequality implies $\limsup_\varepsilon \varphi_s^\varepsilon(x_\varepsilon) \leq C + \frac{C_0}{2r}(2r)^{p_0}H_0(s) < +\infty$ for a.e. $s \in (0,1)$. □

We have not been able to establish φ_s-uniform approximability of K even for very simple functions a. We provide some partial results in Sec. 6. To avoid such a difficulty, we use more flexible estimates below which yield successful completion of the remaining steps. Crucial ingredient is a kind of strong convergence of the lower-order term $A(s,v,v')$. We set $I^0_\alpha(v) := \int_0^1 (W(v'(s)) + \alpha v^2(s))ds$, where $v \in \mathrm{W}^{1,1}(0,1)$, and $A_\infty(s) := \int_{\mathbf{R}^2} A(s,\boldsymbol{\xi})d\nu_\infty(\boldsymbol{\xi})$. Then we obtain the following result.

Proposition 4.4. *If there holds* (4.1)–(4.3) *and if* (v_ε) *is FE sequence for* $(\varepsilon^{-2/3}I^\varepsilon_A)$, *then there holds*:

$$\underline{\delta}_{(v_\varepsilon,v'_\varepsilon)} \overset{*}{\rightharpoonup} \frac{1}{2}\delta_{(0,-1)} + \frac{1}{2}\delta_{(0,1)} \quad \text{in } \mathrm{L}^\infty_{w*}((0,1);\mathcal{P}(\mathbf{R}^2)), \tag{4.12}$$

$$\lim_{\varepsilon\to 0} A(s,v_\varepsilon,v'_\varepsilon) = A_\infty(s) \quad (\text{a.e. } s \in (0,1)). \tag{4.13}$$

Proof. We simply note that every FE sequence for $(\varepsilon^{-2/3}I_A^\varepsilon)$ is also a minimizing sequence for $I_{\alpha_0}^0$. Then it is a well-known fact (cf. [1, p. 763] or [13, p. 36]) that, under assumption (4.1), every minimizing sequence (w_ε) for $I_{\alpha_0}^0$ share the properties $w_\varepsilon \to 0$ in $\mathrm{L}^2(0,1)$, $\underline{\delta}_{w'_\varepsilon} \xrightharpoonup{*} \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ in $\mathrm{L}^\infty_{w*}((0,1);\mathcal{P}(\mathbf{R}))$. Thus, every given subsequence (ε_n) has further subsequence (ε_{n_m}) such that $w_{\varepsilon_{n_m}}(s) \to 0$ for a.e. $s \in (0,1)$ as $m \to \infty$. By [13, Corollary 3.4] we obtain (4.12). To verify (4.13), we use [13, Corollary 3.3], getting $\liminf_{\varepsilon\to 0} A(s, v_\varepsilon, v'_\varepsilon) \geq A_\infty(s)$ for a.e. $s \in (0,1)$ for arbitrary positive Carathéodory function a. On the other hand, by (4.1), (v'_ε) is bounded in $\mathrm{L}^{r_0}(0,1)$ and (by the Rellich imbedding theorem) (v_ε) is bounded in $\mathrm{L}^q(0,1)$ for every $q \in [1,+\infty)$. Therefore, by (4.3), the sequence $\sigma \mapsto a(s,\sigma,v_\varepsilon(\sigma),v'_\varepsilon(\sigma))$ is bounded in $\mathrm{L}^1(0,1)$ for a.e $s \in (0,1)$. Then Chacon's biting lemma (cf. [3]), combined with the fundamental theorem of Young measures, provides

$$\begin{aligned}\limsup_{\varepsilon\to 0} A(s, v_\varepsilon, v'_\varepsilon) &\leq \limsup_{\varepsilon\to 0} \int_{(0,1)\setminus E_j} a(s,\sigma,v_\varepsilon(\sigma),v'_\varepsilon(\sigma))d\sigma + \int_0^1 h_j(s,\sigma)d\sigma \\ &= \int_{(0,1)\setminus E_j} \left(\frac{1}{2}a(s,\sigma,0,-1) + \frac{1}{2}a(s,\sigma,0,1)\right) d\sigma \\ &\quad + \int_0^1 h_j(s,\sigma)d\sigma,\end{aligned}$$

where (E_j) is a sequence of Borel measurable sets such that $\lim_{j\to+\infty} \lambda(E_j) = 0$, $E_j \subseteq (0,1)$, $h_j(s,\sigma) := h_0(s,\sigma)\chi_{E_j}(\sigma) + c_1 h_1(s) + c_2 h_2(s)$, $\|v_\varepsilon\|^q_{\mathrm{L}^q(0,1)} \leq c_1$, $\|v'_\varepsilon\|^p_{\mathrm{L}^p(0,1)} \leq c_2$. At last, we pass to the limit as $j \to +\infty$ in the last inequality. $\square$

Before we present the proof of Γ-convergence result for the relaxed functionals, we introduce some further notation. We define $f^\varepsilon_{s,\infty}, f_{s,\infty} : K \to [0,+\infty]$ by $f^\varepsilon_{s,\infty}(y) := ⨍_{-r}^r (\varepsilon^{2/3}y''^2(\tau) + \varepsilon^{-2/3}W(y'(\tau)) + A_\infty(s)y^2(\tau))d\tau$, if $y \in \mathrm{H}^2(-r,r)$ ($f^\varepsilon_{s,\infty}(y) := +\infty$, otherwise),

$$f_{s,\infty}(y) := \frac{\mathcal{A}_0}{2r}|Sy' \cap (-r,r)| + A_\infty(s) ⨍_{-r}^r y^2(\tau)d\tau, \tag{4.14}$$

if $y \in \mathcal{S}(-r,r)$ ($f_{s,\infty}(y) := +\infty$, otherwise). Then by the theorem of Modica and Mortola (cf. [11]) it follows

$$f^\varepsilon_{s,\infty} \xrightarrow{\Gamma} f_{s,\infty} \quad \text{on } K \text{ (a.e. } s \in (0,1)). \tag{4.15}$$

To proceed, we define $F^\varepsilon_A, F_A : \mathrm{YM}((0,1);K) \to [0,+\infty]$ by

$$F^\varepsilon_A(\boldsymbol{\nu}) := \begin{cases} \displaystyle\int_0^1 \langle \nu_s, \varphi^\varepsilon_s \rangle ds & \text{if } \boldsymbol{\nu} = \underline{\delta}_{R^\varepsilon v} \text{ for some } v \in \mathrm{H}^2_{\mathrm{per}}(0,1), \\ +\infty & \text{otherwise,} \end{cases} \tag{4.16}$$

$$F_A(\boldsymbol{\nu}) := \begin{cases} \displaystyle\int_0^1 \langle \nu_s, \varphi_s \rangle ds & \text{if } \nu_s \in \mathcal{I}(K) \text{ for a.e. } s \in (0,1), \\ +\infty & \text{otherwise.} \end{cases} \tag{4.17}$$

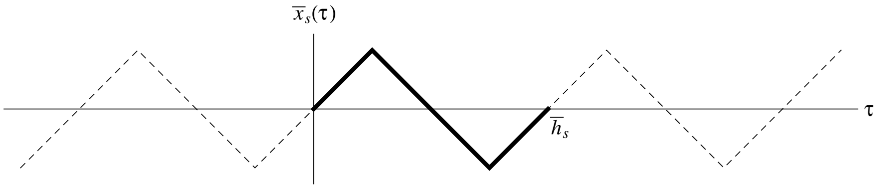

Fig. 1. Sawtooth function $\overline{x}_s \in \mathcal{S}_{\rm per}(0, \overline{h}_s)$, $\overline{h}_s := (\frac{48\mathcal{A}_0}{A_\infty(s)})^{1/3}$.

Accordingly, $F_{A_\infty} : \mathrm{YM}((0,1);K) \to [0,+\infty]$ is defined by $F_{A_\infty}(\boldsymbol{\nu}) := \int_0^1 \langle \nu_s, f_{s,\infty}\rangle ds$, if $\nu_s \in \mathcal{I}(K)$ for a.e. $s \in (0,1)$ ($F_{A_\infty}(\boldsymbol{\nu}) := +\infty$, otherwise).

Theorem 4.5. *If* (4.1)–(4.3) *hold, then* $F_A^\varepsilon \xrightarrow{\Gamma} F_{A_\infty}$ *as* $\varepsilon \to 0$ *on* $\mathrm{YM}((0,1);K)$. *Besides, if* v_ε *minimizes* I_A^ε, *then* (v_ε) *satisfies* $\underline{\delta}_{R^\varepsilon v_\varepsilon} \xrightarrow{*} \mathcal{E}_{\overline{x}}$ *in* $\mathrm{YM}((0,1);K)$ *as* $\varepsilon \to 0$, *where* $\mathcal{E}_{\overline{x}}(s) := \epsilon_{\overline{x}_s}$ *for almost every* $s \in (0,1)$, *with* $\overline{x}_s \in \mathcal{S}_{\rm per}(0, \overline{h}_s)$ *depicted in Fig.* 1.

Proof. First, we obtain the lower bound. If $\boldsymbol{\nu}^\varepsilon \xrightarrow{*} \boldsymbol{\nu}$ in $\mathrm{YM}((0,1);K)$ as $\varepsilon \to 0$, we can assume that there holds $\liminf_\varepsilon F_A^\varepsilon(\boldsymbol{\nu}^\varepsilon) < +\infty$. By (4.16) and by extracting a subsequence (which we do not relabel) such that liminf is actually a limit, for sufficiently small ε we have $\boldsymbol{\nu}^\varepsilon = \underline{\delta}_{R^\varepsilon v_\varepsilon}$, whereby (v_ε) is FE sequence for $(\varepsilon^{-2/3} I_A^\varepsilon)$. Now, by Theorem 3.1, (4.15) and Fatou's lemma, we estimate

$$\begin{aligned}\liminf_{\varepsilon\to 0} F_A^\varepsilon(\underline{\delta}_{R^\varepsilon v_\varepsilon}) &\geq \liminf_{\varepsilon\to 0} F_{A_\infty}^\varepsilon(\underline{\delta}_{R^\varepsilon v_\varepsilon}) + \liminf_{\varepsilon\to 0}\int_0^1 (A_\varepsilon(s) - A_\infty(s))\varepsilon^{-2/3} v_\varepsilon^2(s) ds \\ &\geq F_{A_\infty}(\boldsymbol{\nu}) + \int_0^1 \liminf_{\varepsilon\to 0}(A_\varepsilon(s) - A_\infty(s))\varepsilon^{-2/3} v_\varepsilon^2(s) ds,\end{aligned}$$

where $A_\varepsilon(s) := A(s, v_\varepsilon, v_\varepsilon')$. Then Corollary 3.3 in [13] yields the lower bound. Next, we deal with the upper bound. Let $\boldsymbol{\nu} \in \mathrm{YM}((0,1);K)$ be such that there holds $\nu_s \in \mathcal{I}(K)$ for a.e. $s \in (0,1)$. We claim that there exists a sequence $(\overline{v}_\varepsilon)$ such that there holds $\underline{\delta}_{R^\varepsilon \overline{v}_\varepsilon} \xrightarrow{*} \boldsymbol{\nu}$ in $\mathrm{YM}((0,1);K)$ as $\varepsilon \to 0$ and $\limsup_\varepsilon F_A^\varepsilon(\underline{\delta}_{R^\varepsilon \overline{v}_\varepsilon}) \leq F_{A_\infty}(\boldsymbol{\nu})$. Theorem 3.4 in [1] and (4.15) provide $F_{A_\infty^M}^\varepsilon \xrightarrow{\Gamma} F_{A_\infty^M}$ on $\mathrm{YM}((0,1);K)$ as $\varepsilon \to 0$, where $A_\infty^M(s) := \min\{A_\infty(s), M\}$ and $M > 0$. More precisely, the upper bound is achieved by proving the following property (cf. [1, pp. 788–789]): for every $\eta > 0$ and $M > 0$ there exist $\overline{M}_\eta > 0$ and a sequence $(\overline{v}_\varepsilon)$ with properties

$$\|\overline{v}_\varepsilon\|_{\mathrm{L}^\infty(\mathbf{R})} \leq \overline{M}_\eta \varepsilon^{1/3}, \quad \overline{v}_\varepsilon \in \mathrm{H}^2_{\rm per}(0,1), \quad \mathrm{Lip}(\overline{v}_\varepsilon) \leq 1, \tag{4.18}$$

$$\limsup_{\varepsilon\to 0} \phi(\delta_{R_s^\varepsilon \overline{v}_\varepsilon} - \nu_s) \leq \eta \quad (\text{a.e. } s \in (0,1)), \tag{4.19}$$

$$\limsup_{\varepsilon\to 0} F_{A_\infty^M}^\varepsilon(\underline{\delta}_{R^\varepsilon \overline{v}_\varepsilon}) \leq F_{A_\infty^M}(\boldsymbol{\nu}) + \eta. \tag{4.20}$$

By (4.20) we immediately get

$$\limsup_{\varepsilon\to 0} F_{A_\infty}^\varepsilon(\underline{\delta}_{R^\varepsilon \overline{v}_\varepsilon}) \leq F_{A_\infty}(\boldsymbol{\nu}) + \overline{M}_\eta^2 \int_{T_{A_\infty}^M} A_\infty(s) ds + \eta, \tag{4.21}$$

where $T^M_{A_\infty} := \{s \in (0,1) : A_\infty(s) > M\}$. Hence, $(\overline{v}_\varepsilon)$ is FE sequence for $(\varepsilon^{-2/3} I^\varepsilon_{A_\infty})$ (and therefore minimizing sequence for $I^0_{\alpha_0}$) and there holds

$$\underline{\delta}_{(\overline{v}_\varepsilon, \overline{v}'_\varepsilon)} \xrightarrow{*} \frac{1}{2}\delta_{(0,-1)} + \frac{1}{2}\delta_{(0,1)} \quad \text{in } \mathrm{L}^\infty_{w*}((0,1); \mathcal{P}(\mathbf{R}^2)). \tag{4.22}$$

We note that by (4.18) there holds $|A(s, \overline{v}_\varepsilon, \overline{v}'_\varepsilon) - A_\infty(s)| \le 2H(s)$ for a.e. $s \in (0,1)$, where, by (4.3), we have $H(s) := \int_0^1 \max_{\xi_1 \in [-R,R], \xi_2 \in [-1,1]} a(s,\sigma,\xi_1,\xi_2) d\sigma$ for large enough $R > 0$. At this point we consider function a_R such that $a_R(s,\sigma,\xi_1,\xi_2) = a(s,\sigma,\xi_1,\xi_2)$ for every $(\xi_1,\xi_2) \in [-R,R] \times [-R,R]$ and $a_R(s,\sigma,\cdot,\cdot\cdot) \in \mathrm{C}_0(\mathbf{R}^2)$. By (4.18) for sufficiently small ε and every $\sigma \in \mathbf{R}$ there holds $(\overline{v}_\varepsilon(\sigma), \overline{v}'_\varepsilon(\sigma)) \in [-R,R] \times [-R,R]$ and so $A_R(s, \overline{v}_\varepsilon, \overline{v}'_\varepsilon) = A(s, \overline{v}_\varepsilon, \overline{v}'_\varepsilon)$, where $A_R(s, v, v') := \int_0^1 a_R(s, \sigma, v(\sigma), v'(\sigma)) d\sigma$. By the fundamental theorem of Young measures and by (4.22) there holds $\lim_\varepsilon A(s, \overline{v}_\varepsilon, \overline{v}'_\varepsilon) = \lim_\varepsilon A_R(s, \overline{v}_\varepsilon, \overline{v}'_\varepsilon) = A_\infty(s)$, while the dominated convergence theorem and the fact that $H \in \mathrm{L}^1(0,1)$ imply $\lim_\varepsilon \int_0^1 |A(s, \overline{v}_\varepsilon, \overline{v}'_\varepsilon) - A_\infty(s)| ds = 0$. By (4.18) we get

$$F^\varepsilon_A(\underline{\delta}_{R^\varepsilon \overline{v}_\varepsilon}) \le F^\varepsilon_{A_\infty}(\underline{\delta}_{R^\varepsilon \overline{v}_\varepsilon}) + \overline{M}^2_\eta \int_0^1 |A(s, \overline{v}_\varepsilon, \overline{v}'_\varepsilon) - A_\infty(s)| ds.$$

Consequently, by (4.21) it follows

$$\limsup_{\varepsilon \to 0} F^\varepsilon_A(\underline{\delta}_{R^\varepsilon \overline{v}_\varepsilon}) \le F_{A_\infty}(\boldsymbol{\nu}) + \overline{M}^2_\eta \int_{T^M_{A_\infty}} A_\infty(s) ds + \eta. \tag{4.23}$$

As $M \to +\infty$ and $\eta \to 0$, (4.23) and (4.20) amount to $\limsup_\varepsilon F^\varepsilon_A(\underline{\delta}_{R^\varepsilon \overline{v}_\varepsilon}) \le F_{A_\infty}(\boldsymbol{\nu})$ and $\limsup_\varepsilon \Phi(\underline{\delta}_{R^\varepsilon \overline{v}_\varepsilon} - \boldsymbol{\nu}) = 0$. Regarding the second assertion, by [1, Proposition 5.8] we know K is $f_{s,\infty}$-uniformly approximable for a.e. $s \in (0,1)$, which, in turn, implies that the unique minimizer for F_{A_∞} is $\mathcal{E}_{\overline{x}}$ (cf. [1, Theorem 3.12]). Therefore ε-blowups of the minimizers (v_ε) generate $\mathcal{E}_{\overline{x}}$ as $\varepsilon \to 0$. □

Corollary 4.6. *If there holds* (4.1)–(4.3), *then we have* $\mathcal{E}_A = \mathcal{E}_{A,\mathrm{per}} = E_0 \int_0^1 A^{1/3}_\infty(s) ds$.

Proof. It is enough to prove that there holds $\mathcal{E}_A \ge E_0 \int_0^1 A^{1/3}_\infty(s) ds$. Consider $v_\varepsilon \in \mathrm{H}^2(0,1)$ which minimizes I^ε_A and a sequence of open intervals (ω_n) such that $\omega_n \subset\subset (0,1)$ and $\omega_n \nearrow (0,1)$ as $n \to +\infty$. By Theorems 4.5, 3.1 and Proposition 3.3, we estimate $\liminf_\varepsilon \int_{\omega_n} \varphi^\varepsilon_s(R^\varepsilon_s v_\varepsilon) ds \ge E_0 \int_{\omega_n} A^{1/3}_\infty(s) ds$. Since for $\theta := s + \varepsilon^{1/3}\tau$ there holds

$$⨍_{-r}^{r} \int_{\omega_n} (\varepsilon^2 v''^2_\varepsilon(\theta) + W(v'_\varepsilon(\theta)) + A(\theta, v_\varepsilon, v'_\varepsilon) v^2_\varepsilon(\theta)) ds d\tau = \int_{\omega_n} \varphi^\varepsilon_s(R^\varepsilon_s v_\varepsilon) ds,$$

we derive $\mathcal{E}_A \ge E_0 \int_{\omega_n} A^{1/3}_\infty(s) ds$. At last, we let n tend to infinity. □

Corollary 4.7. *If* (4.1)–(4.3) *hold, and if* $s \mapsto \int_0^1 h_0(s,\sigma) d\sigma \in \mathrm{L}^\infty(0,1)$, $h_1, h_2 \in \mathrm{L}^\infty(0,1)$, *then there holds*: (v_ε) *is FE sequence for* $(\varepsilon^{-2/3} I^\varepsilon_A)$ *if and only if* (v_ε) *is FE sequence for* $(\varepsilon^{-2/3} I^\varepsilon_{A_\infty})$.

Proof. The "only if" part follows from the proof of the lower bound in Theorem 4.5. To verify the "if" part, we note that for any FE sequence (v_ε) for $(\varepsilon^{-2/3}I^\varepsilon_{A_\infty})$ there holds $\varepsilon^{-2/3}I^\varepsilon_A(v_\varepsilon) \leq \varepsilon^{-2/3}I^\varepsilon_{A_\infty}(v_\varepsilon) + \int_0^1 |A(s,v_\varepsilon,v'_\varepsilon) - A_\infty(s)|$ $w_\varepsilon(s)ds$, where $w_\varepsilon(s) := \varepsilon^{-2/3}v^2_\varepsilon(s)$. Then $w_\varepsilon \geq 0$, $w_\varepsilon \in \mathrm{L}^\infty(0,1)$, $\|w_\varepsilon\|_{\mathrm{L}^1(0,1)} \leq c_0$. We estimate $\limsup_\varepsilon \varepsilon^{-2/3}I^\varepsilon_A(v_\varepsilon) \leq C + \limsup_\varepsilon 2\int_0^1 H_0(s)w_\varepsilon(s)ds$, where, by (4.3), for suitable $c_1 > 0$ and $c_2 > 0$ we choose $H_0(s) := \int_0^1 h_0(s,\sigma)d\sigma + c_1h_1(s) + c_2h_2(s) \in \mathrm{L}^\infty(0,1)$. In the end, we apply Hölder's inequality. □

In particular, Corollaries 4.3 and 4.7 show that I^ε_A can be viewed as a lower-order perturbation of $I^\varepsilon_{a_0}$.

5. Identification of the Γ-Limit

In this section we prove that there holds $F_A = F_{A_\infty}$.

Proposition 5.1. *If a satisfies* (4.2) *and* (4.3), *then for every $\nu \in \mathcal{I}(K)$ there holds $\langle \nu, f_{s,\infty}\rangle \geq \langle \nu, \varphi_s\rangle$ for a.e. $s \in (0,1)$, so that $F_{A_\infty} \geq F_A$.*

Proof. Let $s \in (0,1)$ be given. Without loss of generality we can assume that $\nu \in \mathcal{I}(K)$ satisfies $\langle \nu, f_{s,\infty}\rangle < +\infty$. Since K is $f_{s,\infty}$-uniformly approximable, by [1, Corollary 5.11] there exists a sequence (x_k) in K such that $x_k \in \mathcal{S}_{\mathrm{per},0}(0,h_k)$, $\epsilon_{x_k} \xrightarrow{*} \nu$ and $\lim_{k\to+\infty}\langle \epsilon_{x_k}, f_{s,\infty}\rangle = \langle \nu, f_{s,\infty}\rangle$. On the other hand, by the lower-semicontinuity of φ_s and Theorem 3.1 we deduce $\lim_{k\to+\infty}\langle \epsilon_{x_k}, \varphi_s\rangle \geq \langle \nu, \varphi_s\rangle$, getting $\langle \nu, f_{s,\infty}\rangle \geq \langle \nu, \varphi_s\rangle$ for every $\nu \in \mathcal{I}(K)$ and $F_{A_\infty}(\boldsymbol{\nu}) \geq F_A(\boldsymbol{\nu})$ for every $\boldsymbol{\nu} \in \mathrm{YM}((0,1);K)$. □

By Theorem 3.1 for any sequence $(\boldsymbol{\nu}_\varepsilon)$ such that $\boldsymbol{\nu}_\varepsilon \xrightarrow{*} \boldsymbol{\nu}$ in $\mathrm{YM}((0,1);K)$ there holds $\liminf_\varepsilon F^\varepsilon_A(\boldsymbol{\nu}_\varepsilon) \geq F_A(\boldsymbol{\nu})$. One of the consequences of Theorem 4.5 is the conclusion that F_{A_∞} is an optimal lower bound, while F_A is in principle only one of possibly many lower bounds. In the following we establish optimality of F_A thus proving that $F_A = F_{A_\infty}$. To proceed, we introduce the notation $\mathcal{S}_{\mathrm{loc}}(\mathbf{R})$ to denote the set of all functions in K which belong to $\mathcal{S}(-r,r)$ for every $r > 0$. We recall that $\tilde{A}(s,0,\cdot) : K \to [0,+\infty]$ is defined by $\tilde{A}(s,0,y) := \int_0^1 \fint_{-r}^r a(s,\sigma,0,y(t))dtd\sigma$, $y \in K$. Then we can write $\tilde{A}(s,0,x') = (\tilde{A}(s,0,\cdot)\circ D)(x)$, where $D : \mathcal{S}_{\mathrm{loc}}(\mathbf{R}) \to K$ is defined by $D(x)(t) := x'(t)$, $t \in \mathbf{R}$, $x \in \mathcal{S}_{\mathrm{loc}}(\mathbf{R})$, and where x' is distributional derivative of x (therefore $|D(x)(t)| = 1$ for a.e. $t \in \mathbf{R}$). We define $\mathcal{I}_0(K) := \{\nu \in \mathcal{I}(K) : \langle \nu, f_{\alpha_0}\rangle < +\infty\}$. By [1, Corollary 5.11 and Theorem 3.4] (cf. [1, Remark, p. 782]), $\langle \nu, f_{\alpha_0}\rangle$ is independent of r for arbitrary $\nu \in \mathcal{I}(K)$ and so $\mathcal{I}_0(K)$ is independent of r, convex and ϕ-closed (by the lower-semicontinuity of f_{α_0}). We define $D^\# : \mathcal{EI}(K) \cap \mathcal{I}_0(K) \to \mathcal{P}(K)$ by $D^\#(\epsilon_x) := \epsilon_{D(x)}$, where $x \in \mathcal{S}_{\mathrm{per}}(0,h)$ for some $h > 0$. Then we have $D^\#(\epsilon_x) = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ for every $x \in \mathcal{S}_{\mathrm{per}}(0,h)$ and every $h > 0$. Thus $D^\#$ is uniformly ϕ-continuous on $\mathcal{EI}(K) \cap \mathcal{I}_0(K)$, and it can be extended by continuity onto the ϕ-closure of $\mathcal{EI}(K) \cap \mathcal{I}_0(K)$, which (by [1, Corollary 5.11]) equals $\mathcal{I}_0(K)$. Moreover, there holds $D^{\#}(\nu) = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ and $\langle \nu, g \circ D\rangle = \langle D^{\#}\nu, g\rangle$ for every $g \in \mathrm{C}(K)$ and every $\nu \in \mathcal{I}_0(K)$.

Lemma 5.2. *If $\tilde{A}(s,0,\cdot) \in \mathrm{C}(K)$ for a.e. $s \in (0,1)$, then for every $\nu \in \mathcal{I}(K)$ such that $\langle \nu, \varphi_s\rangle < +\infty$ for a.e. $s \in (0,1)$ there holds $\langle \nu, \tilde{A}(s,0,\cdot) \circ D\rangle = A_\infty(s)$ and $\langle \nu, \varphi_s\rangle = \langle \nu, f_{s,\infty}\rangle$ for a.e. $s \in (0,1)$.*

Proof. Consider $\nu \in \mathcal{I}(K)$ such that $\langle \nu, \varphi_s\rangle < +\infty$ for a.e. $s \in (0,1)$. Then by (4.3) there holds $\langle \nu, f_{\alpha_0}\rangle < +\infty$, and we have $f_{\alpha_0}(x) < +\infty$ for ν-a.e. $x \in K$, which gives $x \in \mathcal{S}_{\mathrm{loc}}(\mathbf{R})$ for ν-a.e. $x \in K$. Therefore $\mathcal{S}_{\mathrm{loc}}(\mathbf{R}) \subseteq \mathrm{supp}(\nu)$ for every $\nu \in \mathcal{I}_0(K)$. On the other hand $\langle \nu, f_{\alpha_0}\rangle = \int_{\mathrm{supp}(\nu)} f_{\alpha_0}(x)d\nu(x) < +\infty$ provides $x \in \mathcal{S}_{\mathrm{loc}}(\mathbf{R})$ for ν-a.e. $x \in \mathrm{supp}(\nu)$. Thus there exists a set $E \subset K$ (which depends on ν) such that $E \subset \mathrm{supp}(\nu)$, $\nu(E) = 0$ and $\mathcal{S}_{\mathrm{loc}}(\mathbf{R}) = \mathrm{supp}(\nu)\backslash E$. Hence, $\langle \nu, \tilde{A}(s,0,\cdot) \circ D\rangle$ is well-defined and finite. By [1, Corollary 5.11] there exists a sequence (x_k) in K such that $x_k \in \mathcal{S}_{\mathrm{per},0}(0,h_k)$ and $\epsilon_{x_k} \xrightharpoonup{*} \nu$ as $k \to +\infty$ in $\mathcal{P}(K)$. Continuity of $D^{\#}$ gives $D^{\#}\epsilon_{x_k} \xrightharpoonup{*} D^{\#}\nu$ as $k \to +\infty$ in $\mathcal{P}(K)$, while by assumption $\tilde{A}(s,0,\cdot) \in \mathrm{C}(K)$ it follows $\langle \epsilon_{x_k}, \tilde{A}(s,0,\cdot) \circ D\rangle \to \langle \nu, \tilde{A}(s,0,\cdot) \circ D\rangle$ as $k \to +\infty$. At this point for $x \in \mathcal{S}_{\mathrm{per}}(0,h)$ we calculate

$$\langle \epsilon_x, \tilde{A}(s,0,\cdot) \circ D\rangle = \int_0^1 ⨍_{-r}^{r} ⨍_t^{h-t} a(s,\sigma,0,x'(\xi))d\xi dt d\sigma = A_\infty(s). \tag{5.1}$$

We conclude that for every $k \in \mathbf{N}$ there holds $\langle \epsilon_{x_k}, \tilde{A}(s,0,\cdot) \circ D\rangle = A_\infty(s)$ and $A_\infty(s) = \langle \nu, \tilde{A}(s,0,\cdot) \circ D\rangle$. Therefore there also holds $\langle \nu, \varphi_s\rangle = \langle \nu, f_{s,\infty}\rangle$. □

In the next proposition we show that there are examples of a which satisfy both (4.2) and (4.3) (for instance, if $h_2 = 0$) as well as $\tilde{A}(s,0,\cdot) \in \mathrm{C}(K)$ for a.e. $s \in (0,1)$.

Proposition 5.3. *Suppose that for every $\xi_1, \xi_2 \in \mathbf{R}$ and almost every s and σ there holds*:

$$a(s,\sigma,\xi_1,\xi_2) = b_0(s,\sigma,\xi_1)\frac{2}{\pi}\arctan(\xi_2) + c_0(s,\sigma,\xi_1) \tag{5.2}$$

where b_0, c_0 are nonnegative, measurable in s and σ, continuous in ξ_1, such that $b_0(s,\sigma,\xi_1) + c_0(s,\sigma,\xi_1) \le h_0(s,\sigma) + |\xi_1|^q h_1(s)$, $0 < \alpha_0 \le -b_0(s,\sigma,\xi_1) + c_0(s,\sigma,\xi_1)$. Then $\tilde{A}(s,0,\cdot) \in \mathrm{C}(K)$ for a.e. $s \in (0,1)$, and $\varphi_s^\varepsilon \xrightarrow{\Gamma} \varphi_s$ as $\varepsilon \to 0$ on K for a.e. $s \in (0,1)$. Moreover, there holds $F_A^\varepsilon \xrightarrow{\Gamma} F_A$ as $\varepsilon \to 0$ on $\mathrm{YM}((0,1);K)$.

Proof. Suppose that $y_n \to y$ in K as $n \to +\infty$. By (2.1) we have $\arctan(y_n) \xrightharpoonup{*} \arctan(y)$ in $\mathrm{L}^\infty(\mathbf{R})$. Then for every $r > 0$ it follows $⨍_{-r}^{r} \arctan(y_n(t))dt \to ⨍_{-r}^{r} \arctan(y(t))dt$, so that $\lim_n \tilde{A}(s,0,y_n) = \tilde{A}(s,0,y)$. By (5.2) a satisfies assumptions (4.2)–(4.3). Therefore, by Lemma 4.1, if (x_ε) is FE sequence for (φ_s^ε) which satisfies $x_\varepsilon \to x$ in $\mathrm{W}^{1,1}(-r,r)$ as $\varepsilon \to 0$, then it follows

$A^{\varepsilon}_{s,\tau}(x_\varepsilon) \longrightarrow \tilde{A}(s,0,x')$ in $\mathrm{L}^1(-r,r)$ as $\varepsilon \to 0$ and $\varphi^{\varepsilon}_s \xrightarrow{\Gamma} \varphi_s$ as $\varepsilon \to 0$ on K for a.e. $s \in (0,1)$. The last assertion now follows directly from Lemma 5.2. $\square$

Theorem 5.4. *If a satisfies* (4.2) *and* (4.3), *then $F_A = F_{A_\infty}$. Therefore the sequence (F^{ε}_A) has the commutation property.*

Proof. First, we assume that a is independent of s, whereby we write $\tilde{A}(0,\cdot)$ (respectively, φ) instead of $\tilde{A}(s,0,\cdot)$ (respectively, φ_s). Consider arbitrary $\nu \in \mathcal{I}(K)$ such that $\langle \nu, \varphi\rangle < +\infty$. By Urysohn's lemma there exists a sequence of functions (a^n) which satisfy (5.2) and $(\tilde{A}^n(0,\cdot)\circ D)(x) \to (\tilde{A}(0,\cdot)\circ D)(x)$ for every $x \in \mathcal{S}_{\mathrm{loc}}(\mathbf{R})$ as $n \to +\infty$, where $\tilde{A}^n(0,y) := \int_0^1 ⨍_{-r}^{r} a^n(\sigma,0,y(t))dtd\sigma$. Thus $\int_K (\tilde{A}^n(0,\cdot)\circ D)(x)d\nu(x) \to \int_K(\tilde{A}(0,\cdot)\circ D)(x)d\nu(x)$ and $\tilde{A}^n(0,\cdot) \to \tilde{A}(0,\cdot)$ $D^{\#}\nu$-almost everywhere on K as $n \to +\infty$. Now by Lemma 5.2 there holds $\langle \nu, \tilde{A}(0,\cdot)\circ D\rangle = \langle D^{\#}\nu, \tilde{A}(0,\cdot)\rangle = A_\infty$, and, ultimately, $F_A = F_{A_\infty}$. Next, we assume that a is essentially bounded with respect to s. We consider a sequence of simple functions (a_n) defined by $a_n(s,\sigma,\boldsymbol{\xi}) := \sum_{k=1}^{N_n} a^k_n(\sigma,\boldsymbol{\xi})\chi_{\omega^k_n}(s)$, $(s,\sigma,\boldsymbol{\xi}) \in (0,1)\times(0,1)\times\mathbf{R}^2$, where $a_n \le a$, $a_n \to a$ almost everywhere, $\|a_n\|_{\mathrm{L}^\infty} \le \|a\|_{\mathrm{L}^\infty}$, $\omega^k_n \subseteq (0,1)$ are measurable sets such that $\lambda((0,1)\backslash \bigcup_{k=1}^{N_n}\omega^k_n) = 0$, and $\alpha_0 \le a^k_n \le \|a\|_{\mathrm{L}^\infty}$. In accordance with the notation in Sec. 3, for a measurable set $E \subseteq (0,1)$ we define $F_{A;E}$ as in (4.17), but with $\int_0^1$ replaced by $\int_E$. Then we can write $F_{A_n} = \sum_k F_{A^k_n;\omega^k_n}$, $F_{A_{n,\infty}} = \sum_k F_{A^k_{n,\infty};\omega^k_n}$, where $\tilde{A}^k_n(0,y) := \int_0^1 ⨍_{-r}^{r} a^k_n(\sigma,0,y(t))dtd\sigma$ and $A^k_{n,\infty} := \langle \nu_\infty, \tilde{A}^k_n(0,\cdot)\rangle$. We already proved that there holds $F_{A^k_n;\omega^k_n} = F_{A^k_{n,\infty};\omega^k_n}$. In effect, we have $F_{A_n} = F_{A_{n,\infty}}$, where $\tilde{A}_n(s,0,y) := \int_0^1 ⨍_{-r}^{r} a_n(s,\sigma,0,y(t))dtd\sigma$ and $A_{n,\infty}(s) := \langle \nu_\infty, \tilde{A}_n(s,0,\cdot)\rangle$. Since $\tilde{A}_n(s,0,\cdot) \le \tilde{A}(s,0,\cdot)$ for a.e. $s \in (0,1)$ implies $F_{A_n} \le F_A$, as $n \to \infty$ we get $F_{A_\infty} \le F_A$. Then, by Proposition 5.1, it follows $F_{A_\infty} = F_A$. Indeed, we assume that a is integrable with respect to $s \in (0,1)$. We set $a^M := \min\{a, M\}$, $A^M(s,v,v') := \int_0^1 a^M(s,\sigma,v(\sigma),v'(\sigma))d\sigma$ and $A^M_\infty := \min\{A_\infty, M\}$, where $M > 0$. Then $F_{A^M} \le F_A$ and $F_{A^M} = F_{A^M_\infty}$. Finally, we pass to the limit as $M \to \infty$. $\square$

Corollary 5.5. *For any a which satisfies* (4.2) *and* (4.3) *we have $F^{\varepsilon}_A \xrightarrow{\Gamma} F_A$ as $\varepsilon \to 0$ on* $\mathrm{YM}((0,1);K)$. *In particular, $F_A : \mathrm{YM}((0,1);K) \to [0,+\infty]$ is independent of $r > 0$.*

Proof. The claims follow by Theorems 4.5 and 5.4, and [1, Remark, p. 782]. $\square$

6. On Partial φ_s-Uniform Approximability

In this section we give some sufficient conditions which ensure the commutation property in full generality (as stated in Definition 3.4). We also present some properties which necessarily follow by the commutation property. Our consideration is inspired partly by [1, Theorem 2.12(iv)] (and subsequent remarks therein) and partly by [1, Secs. 4 and 5]. In the following by $\mathrm{Per}(\mathbf{R})$ we denote the set of all periodic functions $x \in K$. If $x \in K$ is h-periodic for some $h > 0$ we write $x \in \mathrm{Per}(0,h)$ (if, in addition, there holds $x(0) = x(h) = 0$, we write $x \in \mathrm{Per}_0(0,h)$).

Definition 6.1. We say that K is uniformly approximable if for every $\varepsilon > 0$ there exists $h = h(\varepsilon) > 0$ such that for every point $x \in K$ we can find $\tilde{x} \in \mathrm{Per}(0,h)$ which satisfies $\fint_0^h d(T_\tau x, T_\tau \tilde{x}) d\tau \le \varepsilon$.

Proposition 6.2. *K is uniformly approximable. Moreover, for every $\nu \in \mathcal{I}(K)$ there exists a sequence (y_k), $y_k \in \mathrm{Per}_0(0,h_k)$, such that $\lim_{k\to+\infty} \phi(\epsilon_{y_k} - \nu) = 0$.*

Proof. By [1, Proposition 5.3] K is uniformly approximable. The proof of the second assertion is obtained as in [1, Proposition 5.8]: by considering $R_h\tilde{x}$ (where, for a given $x \in K$, $\tilde{x} \in K$ is chosen as in Definition 6.1) we can achieve $R_h\tilde{x}(0) = R_h\tilde{x}(h) = 0$, $\fint_0^h d(T_\tau R_h \tilde{x}, T_\tau \tilde{x}) d\tau \le \varepsilon$, and then we continue as in [1, Lemma 5.10 and Corollary 5.11]. □

Next, we introduce the following definition.

Definition 6.3. Consider $f : K \to [0, +\infty]$. We say that K is partially f-uniformly approximable if for every $\varepsilon > 0$ there exists $h = h(\varepsilon) > 0$ such that for every point $x \in K$ we can find $\tilde{x} \in \mathrm{Per}(0,h)$ which satisfies $\fint_0^h f(T_\tau \tilde{x}) d\tau \le \fint_0^h f(T_\tau x) d\tau + \varepsilon$.

Then we obtain the approximation which is a kind of generalization of [1, Corollary 5.11]:

Theorem 6.4. *If K is partially f-uniform approximable, and if $\nu \in \mathcal{I}(K)$ satisfies $\langle \nu, f \rangle < +\infty$, then there exists a sequence (ϵ_{y_k}), $y_k \in \mathrm{Per}_0(0,h_k)$, such that $\lim_{k\to+\infty} \phi(\epsilon_{y_k} - \nu) = 0$, and a sequence (ϵ_{x_k}), $x_k \in \mathrm{Per}(0,\tilde{h}_k)$, such that $\limsup_{k\to+\infty} \langle \epsilon_{x_k}, f \rangle \le \langle \nu, f \rangle$.*

Proof. First assertion follows by Proposition 6.2. To prove the second assertion we fix $\varepsilon > 0$ and $\nu \in \mathcal{I}(K)$ such that $\langle \nu, f \rangle < +\infty$. By [1, Theorem 4.15] there exist $N > 0$, $H > 0$, $z_i \in \mathrm{Per}(0,H)$, $i = 1, \dots, N$, such that $\tilde{\nu} := \sum_{i=1}^N \sigma_i \epsilon_{z_i}$ (where $\sigma_i \in [0,1]$, $\sum_{i=1}^N \sigma_i = 1$) satisfies $\langle \tilde{\nu}, f \rangle \le \langle \nu, f \rangle + \varepsilon$. Therefore $\inf_{\mu \in \mathcal{EI}(K)} \langle \mu, f \rangle \le \langle \nu, f \rangle + \varepsilon$, and there exists $\mu_\varepsilon \in \mathcal{EI}(K)$ such that $\inf_{\mu \in \mathcal{EI}(K)} \langle \mu, f \rangle > \langle \mu_\varepsilon, f \rangle - \varepsilon$. Hence, $\langle \mu_\varepsilon, f \rangle \le \langle \nu, f \rangle + 2\varepsilon$. If $\varepsilon_k \to 0$ as $k \to +\infty$, we obtain $\limsup_k \langle \epsilon_{x_k}, f \rangle \le \langle \nu, f \rangle$, where $\epsilon_{x_k} := \mu_{\varepsilon_k}$. □

For a set $Y \subseteq \mathcal{P}(K)$ by $[Y]'$ we denote the set of all cluster points of Y. For a given $x \in K$ and $h > 0$ we define $\mu_x^h := \fint_0^h \delta_{T_\tau x} d\tau$ and we set $\Delta(x) := \{\mu_x^h : h > 0\}$. Since $\lim_{n\to+\infty} h_n = h \in \mathbf{R}\backslash\{0\}$ implies $\mu_x^{h_n} \xrightarrow{*} \mu_x^h$ in $\mathcal{P}(K)$ as $n \to +\infty$, we deduce that every point of the set $\Delta(x)$ is its cluster point, so that $\Delta(x) \subseteq [\Delta(x)]'$. We set $\mathcal{EI}^\infty(x) := [\Delta(x)]' \cap \mathcal{I}(K)$, $\mathcal{EI}^\infty(K) := \bigcup_{x\in K} \mathcal{EI}^\infty(x)$. Then $\mathcal{EI}^\infty(K)$ is the set of all probability invariant measures "generated" by some $x \in K$. In the following lemma we describe the structure of $\mathcal{EI}^\infty(K)$.

Lemma 6.5. *For every $\tilde{x} \in \mathrm{Per}(\mathbf{R})$ we have $\mathcal{EI}^\infty(\tilde{x}) = \{\epsilon_{\tilde{x}}\}$, and therefore $\mathcal{EI}(K) \subseteq \mathcal{EI}^\infty(K)$. For every $x \in K\backslash\mathrm{Per}(\mathbf{R})$ the set $\mathcal{EI}^\infty(x)$ contains only cluster points of $\Delta(x)$ generated by sequences (h_n) such that $h_n \to +\infty$.*

Proof. If $g \in \mathrm{C}(K)$, we set $g^+ := \max\{g, 0\} \in \mathrm{C}(K)$, $g^- := \min\{g, 0\} \in \mathrm{C}(K)$. Then $g = g^+ + g^-$ and for h_0-periodic $\tilde{x} \in K$ we estimate

$$\begin{aligned} \left\langle \frac{h^*}{h}\epsilon_{\tilde{x}}, g^+ \right\rangle &\leq \langle \mu^h_{\tilde{x}}, g^+\rangle \leq \left\langle \frac{h_*}{h}\epsilon_{\tilde{x}}, g^+ \right\rangle, \\ \left\langle \frac{h_*}{h}\epsilon_{\tilde{x}}, g^- \right\rangle &\leq \langle \mu^h_{\tilde{x}}, g^-\rangle \leq \left\langle \frac{h^*}{h}\epsilon_{\tilde{x}}, g^- \right\rangle, \end{aligned} \tag{6.1}$$

where $h_* := \lceil hh_0^{-1}\rceil h_0$ and $h^* := \lfloor hh_0^{-1}\rfloor h_0$. After summation of the corresponding left- and right-hand sides in (6.1), as we pass to the limit as $h \to +\infty$, we get $\lim_{h\to+\infty} \frac{h^*}{h} = \lim_{h\to+\infty} \frac{h_*}{h} = 1$, $\eta^-_{\tilde{x}} = \eta^+_{\tilde{x}} = \epsilon_{\tilde{x}}$, where $\eta^-_x := \liminf_{h\to+\infty} \mu^h_x$ and $\eta^+_x := \limsup_{h\to+\infty} \mu^h_x$. Next, we consider nonperiodic $x \in K$ and a sequence (h_n) such that $h_n \to 0$. For $\tau \in \mathbf{R}$ we calculate $\langle T^\#_\tau \mu^{h_n}_x - \mu^{h_n}_x, g\rangle = ⨍_\tau^{h_n+\tau} g(T_\xi x)d\xi - ⨍_0^{h_n} g(T_\xi x)d\xi$. An application of L'Hospital's rule implies $\lim_{n\to+\infty}\langle T^\#_\tau \mu^{h_n}_x - \mu^{h_n}_x, g\rangle = g(T_\tau x) - g(x)$. Since for every nonconstant $x \in K$ there exists at least one $\tau \in \mathbf{R}$ such that $T_\tau x \neq x$, by Urysohn's lemma there exists $g \in \mathrm{C}(K)$ such that $g(T_\tau x) \neq g(x)$. This means that measures $\eta^-_{x,0} := \liminf_{h\to 0} \mu^h_x$ and $\eta^+_{x,0} := \limsup_{h\to 0} \mu^h_x$ are not invariant (we similarly deal with the case when the sequence (h_n) converges to h, where $h \in \mathbf{R}\backslash\{0\}$). Finally, we consider the case when $h_n \to +\infty$. Without loss of generality we can assume that (μ^h_x) weakly-star converges to some limit in $\overline{\mu} \in \mathcal{P}(K)$ as $h \to +\infty$ (if necessary we pass to a subsequence). Then for $\tau > 0$ there holds $|\langle T^\#_\tau\overline{\mu} - \overline{\mu}, g\rangle| \leq \lim_{h\to+\infty} \frac{1}{h}(\int_0^\tau |g(T_\xi x)|d\xi + \int_h^{h+\tau} |g(T_\xi x)|d\xi) = 0$, which proves that the measures $\eta^\pm_x$ are invariant. □

For a given function $f : K \to [0, +\infty]$ we define $f^\# : \mathcal{P}(K) \to [0, +\infty]$ by $f^\#(\mu) := \langle \mu, f\rangle$, if $\mu \in \mathcal{I}(K)$ ($f^\#(\mu) := +\infty$, otherwise). By [1, Remark 4.13, p. 803], if f is lower-semicontinuous, so is $f^\#$. We say that $\mathcal{I}(K)$ admits approximation in f-energy at $\nu \in \mathcal{I}(K)$ if there exists a sequence $z_k \in \mathrm{Per}(\mathbf{R})$ such that $\limsup_{k\to+\infty}\langle \epsilon_{z_k}, f\rangle \leq \langle \nu, f\rangle$ and $\lim_{k\to+\infty} \phi(\epsilon_{z_k} - \nu) = 0$. We say that $\mathcal{I}(K)$ admits approximation in f-energy if $\mathcal{I}(K)$ admits approximation in f-energy at every $\nu \in \mathcal{I}(K)$. Now we are ready to state the following result.

Proposition 6.6. *Suppose there holds*:

(i) $\varphi : K \to [0, +\infty]$ *is lower-semicontinuous,*
(ii) $f : K \to [0, +\infty]$ *is lower-semicontinuous,*
(iii) $f^\#(\mu) = \varphi^\#(\mu)$ *for every* $\mu \in \mathcal{EI}(K)$,
(iv) K *is f-uniformly approximable.*

Then the following conclusions hold:

(a) $f^{\#} \geq \varphi^{\#}$.
(b) *If K is partially φ-uniformly approximable, then* $\min \varphi^{\#} = \min f^{\#}$.
(c) *If K is partially φ-uniformly approximable, and if $\varphi^{\#}$ admits an unique minimizer $\overline{\nu}$ on $\mathcal{P}(K)$, then $f^{\#}$ admits an unique minimizer which equals $\overline{\nu}$, and $\mathcal{I}(K)$ admits approximation in φ-energy at $\overline{\nu}$.*
(d) *If* $\inf_{\mu \in \mathcal{EI}^\infty(K)} f^{\#}(\mu) = \inf_{\mu \in \mathcal{EI}^\infty(K)} \varphi^{\#}(\mu)$, *and if* $f^{\#} : \mathcal{P}(K) \to [0, +\infty]$ *admits a minimizer* $\overline{\nu} \in \mathcal{EI}(K)$, *then K is partially φ-uniformly approximable.*
(e) *If there exists $M_0 > 0$ such that K is partially φ^M-uniformly approximable for every $M \geq M_0$, where* $\varphi^M := \min\{\varphi, M\}$, *and if* $f^{\#} : \mathcal{P}(K) \to [0, +\infty]$ *admits a minimizer* $\overline{\nu} \in \mathcal{EI}(K)$, *then* $\inf_{\mu \in \mathcal{EI}^\infty(K)} f^{\#}(\mu) = \inf_{\mu \in \mathcal{EI}^\infty(K)} \varphi^{\#}(\mu)$.

Proof. (a) The claim follows by the lower-semicontinuity of φ and f-uniform approximability of K (as in Proposition 5.1).

(b) By Theorem 6.4 for every $\nu \in \mathcal{I}(K)$ such that $\langle \nu, \varphi \rangle < +\infty$ there exists a sequence (ϵ_{x_k}), $x_k \in \mathrm{Per}(\mathbf{R})$, such that $\limsup_{k \to +\infty} \langle \epsilon_{x_k}, \varphi \rangle \leq \langle \nu, \varphi \rangle$. By weak-star compactness of $\mathcal{P}(K)$ there exists a subsequence (x_{k_j}) and $\mu \in \mathcal{I}(K)$ such that $\epsilon_{x_{k_j}} \xrightarrow{*} \mu$ as $j \to +\infty$. By the lower-semicontinuity of f on K we get $\liminf_{j \to +\infty} \langle \epsilon_{x_{k_j}}, f \rangle \geq \langle \mu, f \rangle$. Therefore by (iii) we have $\varphi^{\#}(\nu) \geq f^{\#}(\mu)$. By the lower-semicontinuity of φ on K we get $\lim_{j \to +\infty} \langle \epsilon_{x_{k_j}}, \varphi \rangle \geq \langle \mu, \varphi \rangle$, so that $\varphi^{\#}(\nu) \geq \varphi^{\#}(\mu)$. If ν minimizes $\varphi^{\#}$, then μ also minimizes $\varphi^{\#}$ and there holds $\min \varphi^{\#} \geq f^{\#}(\mu)$, i.e. $\min \varphi^{\#} \geq \min f^{\#}$. By (a) the converse is also true, and so we have $\min \varphi^{\#} = \min f^{\#}$.

(c) Suppose that the unique minimizer of $\varphi^{\#}$ is $\overline{\nu}$. Consider arbitrary minimizer $\tilde{\nu}$ of $f^{\#}$. By f-uniform approximability of K there exists a sequence (z_n), $z_n \in \mathrm{Per}(\mathbf{R})$, such that $\epsilon_{z_n} \xrightarrow{*} \tilde{\nu}$ as $n \to +\infty$ and $\lim_{n \to +\infty} \langle \epsilon_{z_n}, f \rangle = \langle \tilde{\nu}, f \rangle$. By the lower-semicontinuity of φ on K we get $\lim_{n \to +\infty} \langle \epsilon_{z_n}, \varphi \rangle \geq \langle \tilde{\nu}, \varphi \rangle$. By (iii) it follows $\min f^{\#} = f^{\#}(\tilde{\nu}) \geq \varphi^{\#}(\tilde{\nu}) \geq \min \varphi^{\#}$. By (b) we actually have equalities instead of inequalities in the last formula, which means that $\tilde{\nu}$ also minimizes $\varphi^{\#}$. Thus $\tilde{\nu} = \overline{\nu}$ and $\langle \overline{\nu}, f \rangle = \langle \overline{\nu}, \varphi \rangle$, which gives the desired approximation in φ-energy.

(d) We assume the opposite. Then there exists $\varepsilon_0 > 0$ such that for every $h > 0$ there exists $x_h \in K$ with the following property: for every $\tilde{x} \in \mathrm{Per}(0, h)$ there holds $⨍_0^h \varphi(T_\tau x_h) d\tau < ⨍_0^h \varphi(T_\tau \tilde{x}) d\tau - \varepsilon_0$. We consider $\tilde{h} > 0$ and $\tilde{x} \in \mathrm{Per}(0, \tilde{h})$ such that $\langle \epsilon_{\tilde{x}}, f \rangle = \min\{\langle \nu, f \rangle : \nu \in \mathcal{I}(K)\}$. Then the estimate above holds for every h such that $h\tilde{h}^{-1} \in \mathbf{N}$. By compactness of K there exists a subsequence (x_{h_j}) and $x_\infty \in K$ such that $x_{h_j} \to x_\infty$ in K as $j \to +\infty$, where $\lim_{j \to +\infty} h_j = +\infty$. Moreover, as φ is lower-semicontinuous on K and $T_\tau : K \to K$ is continuous (by [1, Proposition 5.3]), for every $\tau \in \mathbf{R}$ there holds $\liminf_{j \to +\infty} (\varphi \circ T_\tau)(x_{h_j}) \geq (\varphi \circ T_\tau)(x_\infty)$. In effect, by Fatou's lemma we get $\liminf_j \int_{\mathbf{R}} (\frac{1}{h_j}(\varphi(T_\tau x_{h_j}) \chi_{(0,h_j)}(\tau)) - \frac{1}{h_j}(\varphi(T_\tau x_\infty) \chi_{(0,h_j)}(\tau))) d\tau \geq 0$, and, for further subsequence $(x_{h_{j_k}})$, $\lim_k (⨍_0^{h_{j_k}} \varphi(T_\tau x_{h_{j_k}}) d\tau - ⨍_0^{h_{j_k}} \varphi(T_\tau x_\infty) d\tau) \geq 0$. By (iii) it follows

$$\liminf_{k\to+\infty} \fint_0^{h_{j_k}} \varphi(T_\tau x_\infty)d\tau \le \liminf_{k\to+\infty} \fint_0^{h_{j_k}} \varphi(T_\tau x_{h_{j_k}})d\tau \le \min_{\mu\in\mathcal{I}(K)} \langle \mu, f\rangle - \varepsilon_0. \quad (6.2)$$

At this point we observe that for any lower-semicontinuous $\psi : K \to [0, +\infty]$ there holds $\langle \eta_x^-, \psi\rangle \le \psi^-(x)$, where $\psi^-(x) := \liminf_{h\to+\infty} \fint_0^h \psi(T_\tau x)d\tau$, $x \in K$. Indeed, we recall that identity $\langle \eta_x^-, g\rangle = \liminf_{h\to+\infty} \fint_0^h g(T_\tau x)d\tau$ holds not only for $g \in \mathrm{C}(K)$, but also for every bounded Borel function $g : K \to \mathbf{R}$ (cf. [1, pp. 800–801]). Therefore we have $\liminf_{h\to+\infty} \fint_0^h \psi(T_\tau x)d\tau \ge \liminf_{h\to+\infty} \fint_0^h \psi^M(T_\tau x)d\tau = \int_K \psi^M(y)d\eta_x^-(y)$, where $\psi^M := \min\{\psi, M\}$ and $\psi^M \nearrow \psi$ as $M \to +\infty$. As we pass to the limit as $M \to +\infty$ in the last inequality, Fatou's lemma yields the claim. Since the estimate (6.2) can be obtained for arbitrary subsequence (x_{h_n}) of the sequence (x_h), we use the observation above to obtain $\langle \eta_{x_\infty}^-, \varphi\rangle \le \varphi^-(x_\infty) \le \min_{\mu\in\mathcal{I}(K)}\langle \mu, f\rangle - \varepsilon_0$. At last, by Lemma 6.5 we get $\inf_{\mu\in\mathcal{EI}^\infty(K)}\langle \mu, \varphi\rangle \le \inf_{\mu\in\mathcal{EI}^\infty(K)}\langle \mu, f\rangle - \varepsilon_0$.

(e) By (a) there holds $\inf_{\mu\in\mathcal{EI}^\infty(K)} \varphi^\#(\mu) \le \inf_{\mu\in\mathcal{EI}^\infty(K)} f^\#(\mu)$. To prove the reverse inequality, we note that for $\overline{\nu} \in \mathcal{EI}(K)$ which minimizes $f^\#$ on $\mathcal{P}(K)$ by (iii) there holds $\inf_{\mu\in\mathcal{EI}^\infty(K)} f^\#(\mu) \le f^\#(\overline{\nu}) = \varphi^\#(\overline{\nu})$ and for every $\delta > 0$ there exists $x_\delta \in K$ such that $\langle \eta_{x_\delta}^-, \varphi\rangle \le \inf_{\mu\in\mathcal{EI}^\infty(K)}\langle \mu, \varphi\rangle + \delta$. On the other hand, for $M \ge M_0$ by partial φ^M-uniform approximability of K for any $\varepsilon > 0$ we can find $h_\varepsilon > 0$ and $\tilde{x}_\varepsilon \in \mathrm{Per}(0, h_\varepsilon)$ such that $\langle \epsilon_{\tilde{x}_\varepsilon}, \varphi^M\rangle \le \langle \mu_{x_\delta}^{h_\varepsilon}, \varphi^M\rangle + \varepsilon$, where $\mu_{x_\delta}^{h_\varepsilon} := \fint_0^{h_\varepsilon} \delta_{T_\tau x_\delta} d\tau$. Then, we again use (iii) to conclude $\liminf_{\varepsilon\to 0}\langle \epsilon_{\tilde{x}_\varepsilon}, f^M\rangle \le \langle \eta_{x_\delta}^-, \varphi^M\rangle \le \langle \eta_{x_\delta}^-, \varphi\rangle \le \inf_{\mu\in\mathcal{EI}^\infty(K)}\langle \mu, \varphi\rangle + \delta$. Since $f^M(x) \to f(x)$ for every $x \in K$ as $M \to +\infty$, by a generalized version of Fatou's lemma we have $\langle \nu, f\rangle \le \liminf_{M\to+\infty}\langle \epsilon_{\tilde{x}_{\varepsilon_M}}, f^M\rangle \le \inf_{\mu\in\mathcal{EI}^\infty(K)}\langle \mu, \varphi\rangle + \delta$, where $\lim_{M\to+\infty} \varepsilon_M = 0$ and (up to a subsequence) $\epsilon_{\tilde{x}_{\varepsilon_M}} \overset{*}{\rightharpoonup} \nu$ in $\mathcal{I}(K)$ as $M \to +\infty$. Therefore $\langle \overline{\nu}, f\rangle \le \langle \nu, f\rangle \le \inf_{\mu\in\mathcal{EI}^\infty(K)}\langle \mu, \varphi\rangle + \delta$. The assertion follows by arbitrariness of δ. □

Corollary 6.7. *Under the assumptions of Proposition* 6.6 *there holds*: *If* $f^\#$ *admits a minimizer* $\overline{\nu} \in \mathcal{EI}(K)$, *then there holds*:

(i) *K is partially φ-uniformly approximable if and only if* $\min \varphi^\# = \min f^\#$.
(ii) *If there exists $M_0 > 0$ such that K is partially φ^M-uniformly approximable for every $M \ge M_0$, then K is partially φ-uniformly approximable.*

Proof. By Proposition 6.6(b), it suffice to check the "if" part of (i). To this end, we suppose that K is not partially φ-uniformly approximable and that there holds $\min \varphi^\# = \min f^\#$. By Proposition 6.6(d), there exists $\varepsilon_0 > 0$ such that $\inf_{\mu\in\mathcal{EI}^\infty(K)} \varphi^\#(\mu) \le \inf_{\mu\in\mathcal{EI}^\infty(K)} f^\#(\mu) - \varepsilon_0$. This is, however, a clear contradiction, because $\min \varphi^\# \le \inf_{\mu\in\mathcal{EI}^\infty(K)} \varphi^\#(\mu)$ and $\min f^\# = \inf_{\mu\in\mathcal{EI}^\infty(K)} f^\#(\mu)$. To prove (ii), we combine conclusions (e) and (d) in Proposition 6.6. □

Corollary 6.8. *If the assumptions of Proposition* 6.6 *are fulfilled and if* $\varphi^\#$ *admits a minimizer* $\overline{\nu} \in \mathcal{EI}(K)$, *then K is partially φ-uniformly approximable.*

Proof. We use assumption (iii) and Proposition 6.6(a), to conclude that $\overline{\nu} \in \mathcal{EI}(K)$ minimizes $f^{\#}$ and that there holds $\min \varphi^{\#} = \min f^{\#}$. Then the claim follows by Corollary 6.7(i). □

The main result of this section establishes the connection between the commutation property and partial φ-uniform approximability.

Theorem 6.9. *If there exists $f_s : K \to [0, +\infty]$ such that K is f_s-uniformly approximable for a.e. $s \in (0,1)$ and $F^{\varepsilon}_{\varphi^{\varepsilon}} \xrightarrow{\Gamma} F_f$ as $\varepsilon \to 0$ on* $\mathrm{YM}((0,1); K)$, *where $\varphi^{\varepsilon}_s \xrightarrow{\Gamma} \varphi_s$ as $\varepsilon \to 0$ on K for a.e. $s \in (0,1)$, then there holds*:

(i) (***Sufficiency***) *If there holds $\varphi^{\#}_s = f^{\#}_s$ for a.e. $s \in (0,1)$, then the sequence $(F^{\varepsilon}_{\varphi^{\varepsilon}})$ has the commutation property.*

(ii) (***Necessity***) *If the sequence $(F^{\varepsilon}_{\varphi^{\varepsilon}})$ has the commutation property and if F_f admits a minimizer $\overline{\boldsymbol{\nu}}$ such that $\overline{\nu}_s \in \mathcal{EI}(K)$ for a.e. $s \in (0,1)$, then K is partially φ_s-uniformly approximable for a.e. $s \in (0,1)$.*

Proof. Claim (i) follows by assumption (ii) in Proposition 6.6 and the definition of F_φ and F_f. We prove the claim (ii) by assuming the opposite. Then there exists a measurable set $E \subseteq (0,1)$ of positive measure such that, for almost every $s \in E$, K is not partially φ_s-uniformly approximable. Then Proposition 6.6(d) provides that for a given $s \in E$ there exists $\varepsilon_0(s) > 0$ such that there holds $\inf_{\mu \in \mathcal{EI}^\infty(K)} \langle \mu, \varphi_s \rangle \leq \inf_{\mu \in \mathcal{EI}^\infty(K)} \langle \mu, f_s \rangle - \varepsilon_0(s)$. Consequently, $\inf_{\mu \in \mathcal{EI}^\infty(K)} \langle \mu, \varphi_s \rangle < \inf_{\mu \in \mathcal{EI}^\infty(K)} \langle \mu, f_s \rangle$ for a.e. $s \in E$, and $F_{\varphi;E} \neq F_{f;E}$. By the commutation property (respectively, assumption (ii) in Proposition 6.6) we have $F^{\varepsilon}_{\varphi^{\varepsilon}} \xrightarrow{\Gamma} F_\varphi$ (respectively, $F^{\varepsilon}_{\varphi^{\varepsilon}} \xrightarrow{\Gamma} F_f$) as $\varepsilon \to 0$ on $\mathrm{YM}((0,1); K)$, while by Proposition 3.3 it follows that $F^{\varepsilon}_{\varphi^{\varepsilon};E} \xrightarrow{\Gamma} F_{\varphi;E}$ (respectively, $F^{\varepsilon}_{\varphi^{\varepsilon};E} \xrightarrow{\Gamma} F_{f;E}$) as $\varepsilon \to 0$ on $\mathrm{YM}(E; K)$. Therefore $F_{\varphi;E} = F_{f;E}$, which, in turn, recovers partial φ_s-uniform approximability of K for a.e. $s \in (0,1)$. □

The model for the consideration in this section are functionals φ_s and $f_{s,\infty}$ considered in Sec. 4. By Theorems 4.5 and 5.4, functional (4.16) provides an example where $\varphi_s \neq f_s$ and $\varphi^{\#}_s = f^{\#}_s$ for a.e. $s \in (0,1)$, whereby φ_s (respectively, f_s) is defined by (4.11) (respectively, (4.14)). If $A(s,0,-1) \neq A(s,0,1)$, we conjecture that K is not φ_s-uniform approximable, but we have not been able to prove it. However, some conclusions are still available.

Corollary 6.10. *Consider φ_s given by* (4.11). *Then K is partially φ_s-uniformly approximable for a.e $s \in (0,1)$. Besides, for every $\boldsymbol{\nu} \in$* $\mathrm{YM}((0,1); K)$ *such that $\langle \nu_s, \varphi_s \rangle < +\infty$ for a.e. $s \in (0,1)$, there exists a sequence $(\epsilon_{z^s_k})$, $z^s_k \in \mathcal{S}_{\mathrm{per},0}(0, h_k(s))$, such that $\lim_{k \to +\infty} \langle \epsilon_{z^s_k}, \varphi_s \rangle = \langle \nu_s, \varphi_s \rangle$ and $\lim_{k \to +\infty} \phi(\epsilon_{z^s_k} - \nu_s) = 0$ for a.e. $s \in (0,1)$ (i.e. $\mathcal{I}(K)$ admits approximation in φ_s-energy for a.e. $s \in (0,1)$).*

Proof. To apply Theorem 6.9, we note that assumption (i) of Proposition 6.6 holds by Proposition 4.2, while assumptions (ii) and (iv) hold by [1, Theorem 4.3] and [1, Proposition 5.8]. By [1, p. 778], identity (5.1) holds for arbitrary a which satisfies (4.2) and (4.3), and therefore assumption (iii) holds as well. Then the first assertion is an immediate consequence of Theorem 5.4, Theorem 3.12 in [1], and Theorem 6.9(ii). We prove the second assertion by combining [1, Theorem 5.4 and Corollary 5.11]. □

Roughly speaking, the analysis in [1] shows that the following principle holds: f-uniform approximability of K yields approximation in f-energy. If, in addition, conditions (3) and (4) described on [1, p. 783] are fulfilled, Γ-convergence of the relaxed functionals is basically guaranteed (modulo the "gluing" construction in [1, pp. 788–789]), which ensures that assumptions of Theorem 6.9 are fulfilled. By [1, Corollary 6.10 and Lemma 3.8], the set of all piecewise constant Young measures $\boldsymbol{\mu} \in \mathrm{YM}((0,1);K)$ such that $\mu_s \in \mathcal{EI}(K)$ for a.e. $s \in (0,1)$ is F_A-dense in $\mathrm{YM}((0,1);K)$ (cf. [1, Definition 3.7]), and the proof of Γ-convergence of the sequence (F_A^ε) now can be conducted exactly as in [1, Theorem 3.4]. We conclude that it is not necessary to have φ-uniform approximability of K in order to obtain approximation in φ-energy (therefore it is a sufficient, but not a necessary condition for Γ-convergence on $\mathrm{YM}((0,1);K)$). In the last corollary we deduce a further sufficient condition for the commutation property in the general case.

Corollary 6.11. *Suppose that $\varphi_s, f_s : K \to [0,+\infty]$ satisfy the assumptions of Proposition* 6.6 *and Theorem* 6.9 *for a.e. $s \in (0,1)$, and that for a.e. $s \in (0,1)$ there holds: K is partially φ_s-uniformly approximable and $\varphi_s^{\#}$ admits an unique minimizer on $B_\delta(\mu) := \{\nu \in \mathcal{P}(K) : \phi(\mu-\nu) \le \delta\}$ for every $\delta > 0$ and every $\mu \in \mathcal{I}(K)$ such that $\varphi_s^{\#}(\mu) < +\infty$. Then the sequence $(F_{\varphi^\varepsilon}^\varepsilon)$ has the commutation property.*

Proof. We choose $s \in (0,1)$ such that the assumptions hold and we consider $f := f_s$ and $\varphi := \varphi_s$. For arbitrary $\mu \in \mathcal{I}(K)$ and $\varphi^{\#}(\mu) < +\infty$, we define $f_\delta^{\#,\mu}(\nu) := f^{\#}(\nu)$ (respectively, $\varphi_\delta^{\#,\mu}(\nu) := \varphi^{\#}(\nu)$) if $\nu \in B_\delta(\mu)$ and $f_\delta^{\#,\mu}(\nu) := +\infty$ (respectively, $\varphi_\delta^{\#,\mu}(\nu) := +\infty$), otherwise. Then $f_\delta^{\#,\mu}$ and $\varphi_\delta^{\#,\mu}$ are lower-semicontinuous on $\mathcal{P}(K)$. By Proposition 6.6(c), we have $\min_{\nu\in B_\delta(\mu)} \varphi_\delta^{\#,\mu}(\nu) = \min_{\nu\in B_\delta(\mu)} f_\delta^{\#,\mu}(\nu)$. Then there holds $\varphi_\delta^{\#,\mu} \xrightarrow{\Gamma} \varphi_0^{\#,\mu}$ and $f_\delta^{\#,\mu} \xrightarrow{\Gamma} f_0^{\#,\mu}$ as $\delta \to 0$ on $\mathcal{P}(K)$, where $f_0^{\#,\mu}(\nu) := f^{\#}(\nu)$ (respectively, $\varphi_0^{\#,\mu}(\nu) := \varphi^{\#}(\nu)$) if $\nu = \mu$ and $f_0^{\#,\mu}(\nu) := +\infty$ (respectively, $\varphi_0^{\#,\mu}(\nu) := +\infty$), otherwise. Hence, as $\delta \to 0$ we obtain $\min \varphi_0^{\#,\mu}(\nu) = \min f_0^{\#,\mu}(\nu)$, i.e. $f^{\#}(\mu) = \varphi^{\#}(\mu)$. Therefore the assertion follows by Theorem 6.9(i). □

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References

[1] G. Alberti and S. Müller, A new approach to variational problems with multiple scales, *Comm. Pure Appl. Math.* **54** (2001) 761–825.
[2] J. M. Ball, A version of the fundamental theorem for Young measures, in *PDE's and Continuum Models of Phase Transitions*, eds. M. Rascle *et al.*, Lecture Notes in Physics, Vol. 344 (Springer, New York, 1989), pp. 207–215.
[3] J. M. Ball and F. Murat, Remarks on Chacon's biting lemma, *Proc. Amer. Math. Soc.* **107**(3) (1989) 655–663.
[4] X. Chen and Y. Oshita, Periodicity and uniqueness of global minimizers of an energy functional containing a long-range interaction, *SIAM J. Math. Anal.* **37**(4) (2005) 1299–1332.
[5] X. Chen and Y. Oshita, An application of the modular function in nonlocal variational problems, *Arch. Ration. Mech. Anal.* **186**(1) (2007) 109–132.
[6] R. Choksi, Scaling laws in microphase separation of diblock copolymers, *J. Nonlinear Sci.* **11**(3) (2001) 223–236.
[7] R. Choksi and X. Ren, On the derivation of a density functional theory for microphase separation of diblock copolymers. *J. Statist. Phys.* **113**(1–2) (2003) 151–176.
[8] G. DalMaso, *An Introduction to* Γ*-Convergence*, Progress in Nonlinear Differential Equations (Birkhäuser, Boston, 1993).
[9] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics (CRC Press, Boca Raton, 1992).
[10] D. S. Kurtz and C. W. Swartz, *Theories of Integration*, Series in Real Analysis, Vol. 9 (World Scientific Publishing, River Edge, NJ, 2004).
[11] L. Modica and S. Mortola, Un esempio di Γ-convergenca, *Boll. Unione Mat. Ital. B* (*5*) **14** (1977) 285–299.
[12] S. Müller, Singular perturbations as a selection criterion for periodic minimizing sequences, *Calc. Var. Partial Differential Equations* **1**(2) (1993) 169–204.
[13] S. Müller, Variational models for microstructure and phase transitions, Lecture notes, Max Planck Institut für Mathematik in den Naturwissenschaften (1998).
[14] T. Ohta and K. Kawasaki, Equilibrium morphology of block copolymer melts, *Macromolecules* **19** (1986) 2621–2632.
[15] A. Raguž, Relaxation of Ginzburg–Landau functional with 1-Lipschitz penalizing term in one dimension by Young measures on micro-patterns, *Asymptotic Anal.* **41**(3–4) (2005) 331–361.
[16] A. Raguž, A note on calculation of asymptotic energy for Ginzburg–Landau functional with externally imposed lower-order oscillatory term in one dimension, *Boll. Unione Mat. Ital. B* (*8*) **10** (2007) 1125–1142.
[17] A. Raguž, A result in asymptotic analysis for the functional of Ginzburg–Landau type with externally imposed multiple small scales in one dimension, *Glas. Mat. Ser. III* **44**(64) (2009) 401–421.
[18] E. N. Spadaro, Almost periodic pattern formation: Diblock copolymers' functional, preprint.